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Study of a Nonlinear System of Partial Differential Equations Associated with Stratified Fluids in Three Dimensions

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Abstract

In this paper, we will show some mathematical properties for a nonlinear system of partial differential equations, which describe the dynamics of internal motions of an exponentially stratified fluid in three-dimensional space. Basically, we will study the existence and uniqueness of the weak solution for our system of partial differential equations involving the nonlinear advection term on a finite interval.

Keywords: Stratified fluids; Non-viscous fluids; Nonlinear advection term; System of partial differential equations; Sobolev spaces; Galerkin method.
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1. Introduction

By stratified fluids, we mean those whose density varies spatially; this continuous density variation influences the fluid dynamics. Here we consider stratified fluids in bounded domains with a certain regularity at the border, which corresponds to an initial density distribution in a homogeneous gravitational field so that the results obtained here can find an application in models of the atmosphere and the ocean.

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These systems came to the attention of Russian researchers in the article by S. Sobolev ("On a new problem of mathematical physics"), see in [22]; where a system of equations is introduced as a description the dynamics of these fluids. Later, other great mathematicians of the time appeared who continued to develop generalizations of Sobolev's systems, among them Maslennikova, who, along with several of her students, steered studies on the asymptotic behavior of these systems, see in [15], [16], [17], among others.

They also made a complete study for the two-dimensional case in [18], as well as various properties of the system solutions that model the small oscillations of a rotating fluid. For example, Maslennikova and P. P. Kumar also worked on stabilization and limit amplitude problems for inhomogeneous Sobolev systems in [19]. An article that has been of significant help in carrying this investigation; is the one carried out by Maslennikova and A. Giniatoulline, who studied the spectral properties of operators for hydrodynamics systems of a rotating fluid, see in [20]. For more articles related to this type of system, see in [9]. The arrival of these systems was of foremost importance since they are not Kovalevskaya-type systems; they are systems whose solutions are not obtained in terms of higher order derivatives. Another result that we will mention below is related to the Euler equations, which can be consulted in [23]. In that paper, they studied the system given by

$$\begin{cases} \dfrac{\partial v}{\partial t} + (v \cdot \nabla)v + \nabla p = f \text{ in } \Omega \times (0,T), \\ \operatorname{div}(v) = 0 \text{ in } \Omega \times (0,T), \\ v(x,0) = v_0 \text{ in } \Omega, \end{cases} \tag{1}$$

with boundary conditions $u \cdot n = 0$. They got solutions with some nicely regular properties; see, for example, the theorem given on page 12 of [23]. Moreover, one of the authors treated a more regular part of this system, for example, in [12], having success obtaining solutions with some friendly classical properties due to the presence of a dissipative term. Therefore, the properties described in this article are novel and valuable since we do not have the system's regulator.

We will study the existence and uniqueness of the solutions for the following system:

$$\begin{cases} \dfrac{\partial v_1}{\partial t} + v \cdot \nabla v_1 + \dfrac{\partial p}{\partial x_1} = 0, \\ \dfrac{\partial v_2}{\partial t} + v \cdot \nabla v_2 + \dfrac{\partial p}{\partial x_2} = 0, \\ \dfrac{\partial v_3}{\partial t} + g\rho + v \cdot \nabla v_3 + \dfrac{\partial p}{\partial x_3} = 0, \\ \dfrac{\partial \rho}{\partial t} - \dfrac{N^2}{g} v_3 = 0, \\ \dfrac{\partial v_1}{\partial x_1} + \dfrac{\partial v_2}{\partial x_2} + \dfrac{\partial v_3}{\partial x_3} = 0, \end{cases} \tag{2}$$

where $x = (x_1, x_2, x_3)$ denotes the spatial variable, and $v = v(x,t) = (v_1(x,t), v_2(x,t), v_3(x,t))$ denotes the velocity field of the fluid. Here we have N and g positive constants. The last equation for the nonlinear system is because our fluid is incompressible, p denotes the scalar field of the dynamic pressure, and ρ represents the dynamic density. On the other hand, it is admitted that at time $t = 0$, the velocity and the density

$$v(x,0) = v_0(x) \text{ and } \rho(x,0) = \rho_0(x), \tag{3}$$

are known data, we also assume that the following boundary conditions hold at $\partial\Omega$, that is,

$$v\Big|_{\partial\Omega} = 0 \quad \text{and} \quad \rho\Big|_{\partial\Omega} = 0, \tag{4}$$

for $x \in \Omega \subset \mathbb{R}^3$ and $t > 0$, where $\rho_0 = e^{-Nx_3}$ denotes the initial density before the perturbation and Ω is a bounded domain with smooth boundary. For this, we will show that there are functions v and p defined in $(0, T_*)$ satisfying that $v \in \mathbf{L}^\infty((0,T), W^{m,p}(\Omega)^3)$ and $p \in \mathbf{L}^\infty((0,T_*), W^{m+1,p}(\Omega))$, where $T_* < \inf(T; T_1)^3$, and also satisfy equations (2), (3) and (4), then we apply Galerkin's method defining an approximate solution for our system.

$$v^n(x,t) = \sum_{j=1}^{n} g_{jn}(t) w_j(x),$$

which is bounded in $\mathbf{L}^2((0,T); L^2(\Omega)^3)$ and $\mathbf{L}^\infty((0,T_*); \mathbf{H}^m(\Omega)^3)$ for all $T_* < \inf(T, T_0)$ in the same way v_n^t in $\mathbf{L}^\infty((0,T_*); \mathbf{H}^m(\Omega)^3)$ for all $T_* < \inf(T, T_0)$, in this way, we can obtain a local solution of the equation using the limit step by a compactness argument.

We distribute the paper in six sections. Section 1 introduces and describes the problem; later, in Section 2, we show some basic notations to understand the problem. Section 3 presents the equations of motion for the nonlinear system. In section 4, we study the problem using an approximate solution applying Galerkin's method. In section 5, we will show the existence and uniqueness of the solution for the nonlinear system given by (2); finally, section 6 concludes this work.

2. Previous Definitions and Notations

Before starting with the study and analysis of our nonlinear problem, we introduce some previous elements and the necessary notation to understand the dynamics of non-viscous and incompressible stratified fluids in $\mathbb{R}^3$, considered in our paper.

Let Ω be a domain of the space $\mathbb{R}^3$, and let p in $\mathbb{R}$, such that $1 \leq p \leq \infty$. A function $f : \Omega \longrightarrow \mathbb{R}$ (or $\mathbb{C}$), is said to belong to $\mathbf{L}^p(\Omega)$, if f is measurable and the norm

$$\|f\|_{\mathbf{L}^p(\Omega)} = \begin{cases} \left(\displaystyle\int_\Omega |f(x)|^p \, \mathrm{d}x \right)^{1/p} & \text{si } 1 \leq \infty, \\ \\ \operatorname*{ess\,sup}_{x \in \Omega} |f(x)| & \text{si } p = \infty, \end{cases}$$

is finite. The spaces $\mathbf{L}^p(\Omega)$ are Banach spaces, (see [7] and [10]). Furthermore, in the spaces $\mathbf{L}^p(\Omega)$ the Hölder Inequality is fulfilled, which ensures that, for $f \in \mathbf{L}^p(\Omega)$ and $v \in L^q(\Omega)$ with $\dfrac{1}{p} + \dfrac{1}{q} = 1$ for $1 \leq p, q \leq \infty$, it holds:

$$\int_\Omega |f(x)v(x)| \, \mathrm{d}x \leq \|f(x)\|_{\mathbf{L}^p(\Omega)} \cdot \|v(x)\|_{\mathbf{L}^q(\Omega)}.$$

In particular, when we have that $p = 2$, then $\mathbf{L}^2(\Omega)$ is a Hilbert space with scalar product, (see [8])

$$(f, v)_2 = \int_\Omega f(x) \cdot v(x) \, \mathrm{d}x.$$

We remember that $\mathbf{L}^2(\Omega)$ is one of the essential Hilbert spaces in the mathematical analysis since they appear very frequently in the study of partial differential equations, and it is the space where the kinetic energy is automatically well defined.

As the variational form of a mathematical physics problem appears, we cross the Sobolev's spaces, (see [10]), denoted by $W^{m,p}(\Omega)$, and defined as the set of all functions $f(x) \in \mathbf{L}^p(\Omega)$ that have all the generalized derivatives up to the order p, which also belong to $\mathbf{L}^p(\Omega)$, this is,

$$W^{m,p}(\Omega) = \{ f \in \mathbf{L}^p(\Omega) \text{ such that } D^\alpha f \in \mathbf{L}^p(\Omega) \text{ for all } \alpha \in N^n : |\alpha| \leq m \},$$

which is contained in $\mathbf{L}^p(\Omega)$. The associated norm defined in this space is given by

$$\|f\|_{W^{m,p}(\Omega)} = \left(\sum_{|\alpha| \leq m} \|D^\alpha f\|^p_{\mathbf{L}^p(\Omega)} \right)^{1/p},$$

where $D^\alpha f$ is the weak derivate of order α. We also find other types of Sobolev spaces such as $W^{k,p}_0(\Omega)$.

Note that when $p = 2$, we can simply write $\mathbf{H}^m(\Omega)$ and $\mathbf{H}^m_0(\Omega)$ instead of $W^{m,2}(\Omega)$ and $W^{m,2}_0(\Omega)$ respectively, (see for example [1]). Furthermore, remember that when $m = 1$ and $p = 2$, we have that the space $W^{1,2}(\Omega)$ is better known as $\mathbf{H}^1(\Omega)$, since it is a Hilbert space, endowed with the scalar product:

$$(f, g)_{\mathbf{H}^1(\Omega)} = \int_\Omega f(x) \cdot g(x) \, \mathrm{d}x + \int_\Omega \nabla f(x) \cdot \nabla g(x) \, \mathrm{d}x \quad \text{for all} \quad f, g \in \mathbf{H}^1(\Omega),$$

where

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3} \right),$$

and

$$\nabla g = \left(\frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}, \frac{\partial g}{\partial x_3} \right).$$

The norm induced by the previous scalar product is given by

$$\|f\|_{\mathbf{H}^1(\Omega)} = \left(\|f\|^2_{\mathbf{L}^2(\Omega)} + \sum_{i=k}^{3} \left\| \frac{\partial f}{\partial x_i} \right\|^2_{\mathbf{L}^2(\Omega)} \right)^{1/2}.$$

On the other hand, let us denote the space of functions by $\mathcal{D}(\Omega)$ such that $\varphi : \Omega \longrightarrow \mathbb{R}$ of class $C^\infty(\Omega)$ with compact support and by $\mathcal{D}'(\Omega)$ the space of distributions on Ω.

Throughout this paper, we will use the standard notations for Lebesgue and Sobolev spaces as found in [5], in particular the norm in $\mathbf{L}^2(\Omega)$ and the scalar product in $\mathbf{L}^2(\Omega)$ will be represented by $\|\cdot\|$ and $(\cdot,\cdot)$ respectively.

Let us define

$$(u, v) := \int_\Omega \sum_{j=1}^{3} u_j \cdot v_j \, \mathrm{d}x, \quad u = (u_1, u_2, u_3), \quad v = (v_1, v_2, v_3) \in \mathbf{L}^2(\Omega)^3,$$

$$((u, v)) := \int_\Omega \sum_{j=1}^{3} \nabla u_j \cdot \nabla v_j \, \mathrm{d}x, \quad u = (u_1, u_2, u_3), \quad v = (v_1, v_2, v_3) \in \mathbf{H}^1_0(\Omega)^3,$$

and the associated norms are given from $| u |^2 := (u, u)$ and $\|u\|^2 := ((u, u))$.

Consider the following notation for the solenoidal Banach spaces $\mathbf{H}$ and $\mathbf{V}$, which intrinsically satisfy the condition $\nabla \cdot v = 0$, and which we can represent as:

$$\mathbf{H} = \{v \in L^2(\Omega)^3 : \nabla \cdot v = 0 \text{ in } \Omega; \ \ \gamma_n v = 0 \text{ on } \partial\Omega\},$$

and

$$\mathbf{V} = \{v \in H^1_0(\Omega)^3 : \nabla \cdot v = 0 \text{ in } \Omega\}. \tag{5}$$

Here, $\nabla \cdot v$ denotes the divergence of v and let γ_n denote the normal component of the trace operator, where $\gamma_n : v \longmapsto n \cdot v\Big|_{\partial\Omega} = 0$, here n denotes the external normal to the boundary.

These spaces are frequently used in equations that model the dynamics of inviscid and incompressible stratified fluids in $\mathbb{R}^3$ and are defined as the closure of Θ in $\mathbf{L}^2(\Omega)^3$ and of Θ in $\mathbf{H}^1_0(\Omega)^3$ respectively, where

$$\Theta = \{v \in \mathcal{D}(\Omega)^3 : \nabla \cdot v = 0 \;\text{ in }\; \Omega\}.$$

It is well-known that $\mathbf{H}$ and $\mathbf{V}$ are Hilbert spaces with the scalar product $(\cdot,\cdot)$ and $\|\cdot\|$ respectively.

The spectral theorem [11] is the key to the theory we are developing. This theorem tells us that every self-adjoint compact operator is diagonalizable.

Theorem 2.1 (**Spectral Theorem**).
Let $T \in \mathcal{L}(H)$ be a compact self-adjoint operator. Then, there is an orthonormal system $\{x_n\}$ of eigenvectors of T and its corresponding sequence of eigenvalues λ_n such that for each $x \in H$, we have that

$$Tx = \sum_n \lambda_n \langle x, x_n\rangle x_n.$$

The sequence λ_n is decreasing and if it is infinity converges to 0.

Now, let's recall some classical results that we will need apply Galerkin's method later.

Theorem 2.2 (**[13], Riesz Theorem**).
Every bounded linear functional f on a Hilbert space H can be represented in terms of the inner product, namely,

$$f(x) = \langle x, y\rangle \text{ with } x \in H,$$

where y depends on f, is uniquely determined by f and has norm

$$\|y\| = \|f\|.$$

Theorem 2.3 (**[10], Rellich-Kondrachov Theorem**).
Let U be an open and bounded subset on $\mathbb{R}^n$ and ∂U is C^1. If $(u_m)_{m\in\mathbb{N}}$ is a sequence in $\mathbf{H}^1(U)$ with $u_m \longrightarrow u$, then $u_m \longrightarrow u$ in $\mathbf{L}^2(U)$. In particular, by weak compactness any sequence in $\mathbf{H}^1(U)$ has a subsequence that is convergent in $\mathbf{L}^2(U)$.

3. Equations of Motion for the Nonlinear System

Let us consider a non-viscous fluid that occupies a region $\Omega \subset \mathbb{R}^3$ subject only to the action of gravity with boundary $\Sigma = \partial\Omega \times (0,T)$, smooth enough (at least Lipschitz continuous) and we define $Q_T := \Omega \times (0,T)$ as the domain of our model where the motion of the fluid takes place, (see in [4]). In this case, $T > 0$; $(0,T)$ is the time interval and $t \in (0,T)$ is the temporal variable.
The equations describing the fluid's motion at $\Omega \subset \mathbb{R}^3$ are

$$a = \frac{\partial v}{\partial t} + (v \cdot \nabla)v \tag{6}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho \;\mathrm{d}V = 0. \tag{7}$$

Let us note that

$$\int_V \left(\frac{\partial \rho}{\partial t} + v\nabla\rho + \rho\nabla v\right)\,\mathrm{d}V = 0. \tag{8}$$

By applying Newton's second law, we obtain an equation for the forces acting on the fluid, which is

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho v \, \mathrm{d}V = F \ + \ F_S. \tag{9}$$

Where $F = \displaystyle\int_V \rho f \, \mathrm{d}V$, is the external force, $F_S = -\displaystyle\int_S pn \, \mathrm{d}S$, is the force on the surface and f is the given force per unit mass.
After using the divergence theorem, we get

$$F_S = -\int_S pn \, \mathrm{d}S = -\int_V \nabla p \, \mathrm{d}V. \tag{10}$$

As a consequence of (10), the equation (9) becomes

$$\int_V \left[\frac{\mathrm{d}}{\mathrm{d}t}(\rho v) + \rho v(\nabla v)\right] \mathrm{d}V = \int_V [\rho f - \nabla p] \ \mathrm{d}V. \tag{11}$$

Keeping in mind the equation, (8), and the relation

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho v) + \rho v(\nabla v) = \rho\frac{\mathrm{d}v}{\mathrm{d}t} + v\left(\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho\nabla v\right),$$

the left hand side of (11) is transformed in

$$\int_V \rho\frac{\mathrm{d}v}{\mathrm{d}t} = \int_V [\rho f - \nabla p] \ \mathrm{d}V. \tag{12}$$

Given the arbitrary nature of V and the fact $\dfrac{\mathrm{d}v}{\mathrm{d}t} = \dfrac{\partial v}{\partial t} + (v\cdot\nabla)v$, we obtain that

$$\frac{\partial v}{\partial t} + (v\cdot\nabla)v + \frac{\nabla p}{\rho} = f. \tag{13}$$

Now, if we incorporate the gravitational force acting vertically, and the Coriolis force in the equation (13) $-2\Omega\times v$; emerging in a situation where a coordinate system rotates at the angular frequency Ω such as the case of the Earth's rotation, results in the following outcome.

$$\frac{\partial v}{\partial t} + (v\cdot\nabla)v = -\frac{\nabla p}{\rho} - g\nabla z - 2\Omega\times v. \tag{14}$$

On the other hand, we derive the equation of state $p = p(\rho, s)$, we know that

$$\frac{\mathrm{d}p}{\mathrm{d}t} = c^2\frac{\mathrm{d}\rho}{\mathrm{d}t}, \tag{15}$$

where $c^2 = \dfrac{\partial p}{\partial \rho}$. For more information (see [4], Part II Fluid Mechanics).

When contemplating an incompressible fluid, it becomes apparent that the following equation serves as its representative model.

$$\frac{\partial \rho}{\mathrm{d}t} + \nabla(\rho v) = 0. \tag{16}$$

If the earth's rotation is not considered, we have that $\Omega\times v = 0$. Therefore, we have that $\nabla p_0 = g\rho_0 e_0$.

By downplaying the significance of the nonlinear terms involving v, p', and ρ', we arrive at the subsequent model that warrants investigation. We start from the fact that $\rho = \rho_0 + \rho'$ and $p = p_0 + p'$ thus $\dfrac{\partial \rho}{\partial t} = \dfrac{\partial \rho'}{\partial t}$ and $\nabla p = \nabla p_0 + \nabla p'$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}(p) = \frac{\partial}{\partial t}(p) + v\nabla p = \frac{\partial}{\partial t}(p') + v\nabla p_0 + v\nabla(p').$$

In this way, we obtain the following approximation

$$\frac{\mathrm{d}}{\mathrm{d}t}(p) \cong \frac{\partial}{\partial t}(p') + v\nabla p_0.$$

Now, we take

$$\begin{aligned}
\nabla(\rho v) &= \nabla((\rho_0 + \rho')v) \\
&\cong \nabla(\rho_0 v) \\
&= \nabla \rho_0 v + \rho_0 \ \mathrm{div}\ (v) \\
\frac{\mathrm{d}\rho}{dt} &= \frac{\partial}{\partial t}(\rho_0 + \rho') + v\nabla(\rho_0 + \rho') \\
&= \frac{\partial}{\partial t}\rho' + v\nabla\rho_0 + v\nabla\rho' \\
&= \frac{\partial}{\partial t}\rho' + v\nabla\rho_0.
\end{aligned}$$

Therefore,

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{\partial}{\partial t}\rho' + \frac{\partial}{\partial x_3}\rho_0 v_3.$$

On the other hand, we have

$$\frac{1}{\rho} = \frac{1}{\rho_0 + \rho'} = \frac{1}{\rho_0\left(1 + \dfrac{\rho'}{\rho_0}\right)} \cong \frac{1}{\rho_0}\left(1 - \frac{\rho'}{\rho_0}\right).$$

Then, we calculate the following quotient

$$\begin{aligned}
\frac{\nabla p}{\rho} &= \frac{\nabla p_0 + \nabla p'}{\rho_0\left(1 + \dfrac{\rho'}{\rho_0}\right)} \\
&= \frac{(\nabla p_0 + \nabla p')}{\rho_0}\left(1 - \frac{\rho'}{\rho_0}\right) \\
&= \frac{1}{\rho_0}\nabla p_0 - \frac{1}{\rho_0}\frac{\nabla p_0}{\rho_0}\rho' + \frac{\nabla p'}{\rho_0} - \frac{\nabla p'}{\rho_0}\cdot\frac{\rho'}{\rho_0} \\
&\cong \frac{1}{\rho_0}\nabla p_0 + \frac{1}{\rho_0}\frac{\nabla p_0}{\rho_0}\rho' - \frac{\nabla p'}{\rho_0}.
\end{aligned}$$

Hence,

$$-\frac{\nabla p}{\rho} - g\nabla z = -\frac{\nabla p'}{\rho_0} + \frac{1}{\rho_0}\frac{\nabla p_0}{\rho_0}\rho'. \tag{17}$$

So, the equation (6) becomes

$$\frac{\partial v}{\partial t}+(v\cdot\nabla)v=-\frac{\nabla p'}{\rho_0}+\frac{1}{\rho_0}\frac{\nabla p_0}{\rho_0}\rho'=-\frac{\nabla p'}{\rho_0}+\frac{1}{\rho_0}(-ge_3)\rho',$$

in consequence

$$\frac{\partial v}{\partial t}+(v\cdot\nabla)v+\frac{\nabla p'}{\rho_0}+\frac{g\rho'}{\rho_0}e_3=0. \tag{18}$$

The equation (16) reduces to

$$\frac{\partial}{\partial t}\rho'+\nabla\rho_0 v+\rho_0\ \mathrm{div}(v)=0\Longrightarrow\frac{\partial}{\partial t}\rho'+\frac{\partial}{\partial x_3}\rho_0 v_3+\rho_0\ \mathrm{div}(v)=0.$$

Now, since

$$\frac{\partial}{dt}(p')+v\nabla p_0=c^2\left(\frac{\partial}{\partial t}\rho'+\frac{\partial}{\partial x_3}\rho_0 v_3\right),$$

then

$$\begin{aligned}\frac{\partial}{\partial t}\rho'&=\frac{1}{c^2}\frac{\partial}{dt}p'+\frac{1}{c^2}v\nabla p_0-\frac{\partial}{\partial x_3}\rho_0 v_3\\&=\frac{1}{c^2}\frac{\partial}{dt}p'-\frac{1}{c^2}g\rho_0 v_3-\frac{\partial}{\partial x_3}\rho_0 v_3\\&=\frac{1}{c^2}\frac{\partial}{dt}p'+\frac{\rho_0}{g}\left(-\frac{g^2}{c^2}-\frac{g}{\rho_0}\frac{\partial}{\partial x_3}\rho_0\right)v_3.\end{aligned}$$

In this way, it follows that

$$\frac{\partial}{\partial t}\rho'=\frac{1}{c^2}\frac{\partial}{dt}p'+\frac{\rho_0}{g}N^2(t)v_3, \tag{19}$$

where $N^2(t)=-g\left(\rho_0^{-1}\frac{\partial}{\partial x_3}\rho_0+\frac{g}{c^2}\right)$.

Note that if $c=\infty$, we obtain the following nonlinear system of partial differential equations

$$\begin{cases}\frac{\partial v}{\partial t}+(v\cdot\nabla)v+\frac{\nabla p'}{\rho_0}+\frac{gp'}{\rho_0}e_3=0\\ \mathrm{div}(v)=0\\ \frac{\partial}{\partial t}\rho'=\frac{\rho_0}{g}N^2(t)v_3,\end{cases} \tag{20}$$

in which $N^2(t)=-g\left(\rho_0^{-1}\frac{\partial}{\partial x_3}\rho_0\right)$.

4. Qualitative Analysis for the Nonlinear System

Studying the dynamics of fluids with exponential stratification, such as the ocean and atmosphere, is essential. This understanding is crucial if we want to create a machine to purify the air circulating in various locations within a city. Similarly, the same knowledge is necessary for situations related to the ocean.
We examine fluids that are stratified and have varying densities based on height, following an exponential pattern. This characteristic influences the fluid's dynamics. Although physicists, mechanical and mechanical engineers have studied these types of fluids, they have mostly focused on the linear model and the two-dimensional scenario. There is limited information on the velocity field when nonlinear terms are present

in the model and viscosity is absent. In this section, we will study the following model that describes the dynamics of an exponentially stratified fluid in the absence of viscosity.

We continue with the study of our system of nonlinear partial differential equations given by equation (2). Note that we can represent it in vector form as follows

$$\begin{cases} \dfrac{\partial v}{\partial t} + (v \cdot \nabla)v + \nabla p + g\rho e_3 = 0, \\ \dfrac{\partial \rho}{\partial t} - \dfrac{N^2}{g} v e_3 = 0, \\ \operatorname{div}(v) = 0, \end{cases} \tag{21}$$

We associate the previous system (21) with the initial conditions

$$v(x,0) = v_0(x) \quad ; \quad \rho(x,0) = \rho_0(x), \tag{22}$$

and the Dirichlet boundary conditions

$$v\Big|_{\partial\Omega} = 0 \;\; y \;\; \rho\Big|_{\partial\Omega} = 0, \tag{23}$$

for $x \in \Omega \subset \mathbb{R}^3$ and $t > 0$, where $\rho_0 = e^{-Nx_3}$ denotes the initial density before the perturbation and Ω is a bounded domain with smooth boundary.

We introduce the following definition of a weak solution

Definition 4.1 (Weak Solution).
Let $0 < T < \infty$. As a weak solution for the system (21) to (23), we understand the pair of functions (v, ρ) such that $\rho(t,x) \in \mathbf{L}^\infty((0,T);\mathbf{L}^2(\Omega))$ and $v(t,x) \in \mathbf{L}^\infty((0,T);\mathbf{H}) \cap \mathbf{L}^2((0,T);\mathbf{V})$ such that the following identities are satisfied

$$\int_0^T \{\langle v, \phi_t\rangle + \langle v, (v\cdot\nabla)\phi\rangle - g\langle \rho e_3, \phi\rangle\} \; \mathrm{d}t + \langle v(0), \phi(0)\rangle = 0, \tag{24}$$

and

$$\int_0^T \langle \rho, \varphi_t\rangle + \frac{N^2}{g}\langle v\cdot e_3, \varphi\rangle \mathrm{d}t + \langle \rho_0(x), \varphi(0)\rangle = 0, \tag{25}$$

for all $\phi \in C^1((0,T);\mathbf{V})$ and $\varphi \in C^1((0,T);\mathbf{L}^2(\Omega))$ with $\phi(T) = 0$ and $\varphi(T) = 0$.

As motivation for this generalized solution definition, let us note that these identities are satisfied for smooth solutions, as we can see below.

Indeed:
Let v, ϕ be functions such that each of the terms that appear makes sense, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle v, \phi\rangle = \langle v_t, \phi\rangle + \langle v, \phi_t\rangle.$$

Now, integrating from 0 to T we get:

$$\langle v(T), \phi(T)\rangle - \langle v(0), \phi(0)\rangle = \int_0^T \{\langle v_t, \phi\rangle + \langle v, \phi_t\rangle\} \; \mathrm{d}t.$$

On the other hand, if ϕ is such that $\phi(T) = 0$, it follows that

$$-\int_0^T \langle v_t, \phi\rangle \; \mathrm{d}t = \int_0^T \langle v, \phi_t\rangle \; \mathrm{d}t + \langle v(0), \phi(0)\rangle. \tag{26}$$

Now, using integration by parts, we can see that

$$-\langle \Delta v, \phi \rangle = \langle \nabla v, \nabla \phi \rangle. \tag{27}$$

Similarly, using integration by parts, we can see that

$$-\langle (v \cdot \nabla) v, \phi \rangle = \langle v, (v \cdot \nabla)\phi \rangle. \tag{28}$$

Indeed:

$$\langle (v \cdot \nabla), \phi \rangle = \langle v \cdot \nabla v_1, \phi_1 \rangle + \langle v \cdot \nabla v_2, \phi_2 \rangle + \langle v \cdot \nabla v_3, \phi_3 \rangle,$$

Now,

$$\begin{aligned}
\langle v \cdot \nabla v_i, \phi_i \rangle &= \sum_{j=1}^{3} \left\langle v_j \frac{\partial v_i}{\partial x_j}, \phi_i \right\rangle \\
&= \sum_{j=1}^{3} \int_\Omega (v_j \phi_i) \frac{\partial v_i}{\partial x_j} \, \mathrm{d}x \\
&= -\sum_{j=1}^{3} \int_\Omega v_i \frac{\partial (v_j \phi_i)}{\partial x_j} \, \mathrm{d}x + \overbrace{\text{boundary terms}}^{=0} \\
&= -\langle v_i, \operatorname{div}(\phi_i v) \rangle \\
&= -\langle v_i, v \cdot \nabla \phi_i + \phi_i \operatorname{div}(v) \rangle \\
&= -\langle v_i, v \cdot \nabla \phi_i \rangle.
\end{aligned}$$

In this way, if we add in i for $i = 1$ to $i = 3$ we obtain (28).

Similarly from (26) we get that:

$$-\int_0^T \left\langle \frac{d}{\mathrm{d}t} \rho, \varphi \right\rangle \, \mathrm{d}t = \int_0^T \langle \rho, \varphi_t \rangle \, \mathrm{d}t + \langle \rho_0(x), \varphi(0) \rangle, \tag{29}$$

for all φ such that $\varphi(T) = 0$.

Thus, if (24) and (25) hold, then it follows that (26), (27), (28) and (29) imply that

$$\int_0^T \langle v_t + (v \cdot \nabla) v + g\rho e_3, \phi \rangle \, \mathrm{d}t = 0,$$

and

$$\int_0^T \left\langle \frac{d\rho}{\mathrm{d}t} - \frac{N^2}{g} v \cdot e_3, \varphi \right\rangle \, \mathrm{d}t = 0.$$

Therefore, there exists a function $p(x, t)$ such that

$$v_t + (v \cdot \nabla) v + g\rho e_3 = -\nabla p(x, t)$$

and

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} - \frac{N^2}{g} v_3 = 0.$$

The primary goal of our paper is to demonstrate a theorem of existence and uniqueness of solutions. The solutions of the theorem will be obtained through limits of approximate solutions, so we will first offer a section on creating these solutions using Galerkin's approach. Galerkin's method is a mathematical technique for solving variational problems in infinite-dimensional spaces. By approximating the problem in a sequence of finite-dimensional subspaces, we can find a solution that is easier to study. Once we have solved the problem in this type of space, we can extend it to infinite-dimensional spaces using limits, ultimately constructing a solution to the initial problem. For more information, (see [14]).

Along with its theoretical significance, Galerkin's method also offers a practical approximation approach that will prove vital in our subsequent analysis.

For our purpose, let m fixed, consider the space $\mathbf{X}_m \subset \mathbf{H}^m(\Omega)^3$, where

$$\mathbf{X}_m = \{v \in \mathbf{H}^m(\Omega)^3 : \nabla \cdot v = 0 \text{ and } v\Big|_{\partial\Omega} = 0\}, \tag{30}$$

endowed with the inner product in a Hilbert space $\langle \cdot, \cdot \rangle_m$.

4.1. Construction of Approximate Solutions

We shall use Galerkin's approach to demonstrate the existence of a solution. But first, we will build a Hilbert basis $(w_k)_{k\in\mathbb{N}}$ of the space $\mathbf{X}_m \subset \mathbf{H}^m(\Omega)$, which we will describe later. Then, with the basis selected, Galerkin's approach will help us in projecting our system into the subspaces $\mathbf{H}^m(\Omega)$, which are generated by the vectors $w_1, w_2, \ldots w_N$ for $N \in \mathbb{N}$. As a result, we have an ordinary differential equations system defined in Euclidean space that has a solution on some interval $[0, T_N]$, as guaranteed by the Cauchy-Lipschitz theorem. Consequently, we must ensure that the T_N are not dependent on N and that the solutions are additionally uniformly bounded in $\mathbf{H}^m(\Omega)$.

Finally, we will apply a compactness argument to derive a local solution for the system passing to the limit.

It can be observed from the equation (30) that when m is equal to zero, the space $\mathbf{X}_0$ is obtained, which is a closed subspace of $\mathbf{L}^2(\Omega)^3$ due to the continuity of the divergence and the trace operators. Additionally, it is true that $\mathbf{X}_m$ is a subset of $\mathbf{X}_0$.

According to the Riesz Theorem given by (2.2), for each $g \in \mathbf{X}_0$, there exists a corresponding $w \in \mathbf{X}_m$ that satisfies the condition

$$\langle w, v \rangle_m = \langle g, v \rangle.$$

If u and v belong to $\mathbf{X}_0$, then there are unique $w(u)$ and $w(v)$ in $\mathbf{X}_m$ such that:

$$\langle w(u), g \rangle_m = \langle u, g \rangle \text{ for all } g \in \mathbf{X}_m$$

and

$$\langle w(v), h \rangle_m = \langle v, h \rangle \text{ for all } h \in \mathbf{X}_m.$$

Therefore, we have

$$\langle w(u), w(v) \rangle_m = \langle u, w(v) \rangle \text{ and } \langle w(v), w(u) \rangle_m = \langle v, w(u) \rangle.$$

The last equation implies

$$\langle w(v), u \rangle = \langle v, w(u) \rangle. \tag{31}$$

Consequently, the equation (31) shows that the operator we defined above, $w : g \longrightarrow w(g)$, is a self-adjoint operator on $\mathbf{X}_0$.

Taking the compact injection $\iota : \mathbf{X}_m \longrightarrow \mathbf{X}_0$ and applying the Rellich-Kondrachov Theorem given by (2.3), we may conclude that the operator.

$$\begin{aligned} w : \mathbf{X}_0 &\longrightarrow \mathbf{X}_0 \\ g &\longmapsto w(g), \end{aligned} \tag{32}$$

is also compact (taking the composition between the identity map and w, considering that the composition between a continuous linear operator and a compact operator is also compact).

Because w is a self-adjoint and compact operator (using the spectral theorem for self-adjoint compact operators), it has a complete orthonormal family of eigenvectors w_k, with $w_k \in \mathbf{X}_m$ such that

$$\langle w_k, v\rangle_m = \lambda_k \langle w_k, v\rangle \text{ for all } v \in \mathbf{X}_m. \tag{33}$$

We can now use Galerkin's technique and the base we found in (33), to get an approximate solution to our problem. For a natural number n, we need to find solutions for system (2) of the form:

$$v^n(x,t) = \sum_{j=1}^{n} g_{jn}(t) w_j(x).$$

where the coefficients g_{jn} in $C^1[0,T]$ are unknown. The expression $\rho^n(x,t)$ can be written without loss of generality as:

$$\rho_t^n(x,t) = \frac{N^2}{g} v^n \cdot e_3 \quad \textit{and} \quad \rho^n\Big|_{t=0} = \rho^0(x). \tag{34}$$

For the more general case, $\rho^0(x)$ we can rewrite it as the limit of a sequence $(\rho_n^0(x)) \subset \mathbf{C}^1(\Omega)$ and we will solve the initial value problem given by (34).

Our objective is to demonstrate the existence of functions called $g_{jn}(t)$.

To do so, we can substitute $v^n(x,t)$ in (24) and select $\phi(x,t) = H(t)w_k(x)$, where $H(t) \in \mathbf{C}^1([0,T])$ and $H(T) = 0$. This yields the following equation:

$$\int_0^T \{\langle v_t^n + (v^n \cdot \nabla) v^n + g\rho^n e_3, w_k\rangle\} H(t)\ \mathrm{dt} = 0.$$

Since $H(t)$ is arbitrary, we can obtain the approximate system for $k = 1, 2, \ldots\ldots, n$:

$$\langle v_t^n + (v^n \cdot \nabla) v^n + g\rho^n e_3, w_k\rangle = 0. \tag{35}$$

We can express equation (35) using the base provided in (33).

$$\begin{aligned} \left(\frac{\mathrm{d}}{\mathrm{dt}}\right) \langle v^n, w_k\rangle + ((v^n \cdot \nabla) v^n, w_k) &= \langle -g\rho^n e_3, w_k\rangle, \\ v^n(0) = v_{0n} &= P_n V_0. \end{aligned} \tag{36}$$

Where P_n represents the orthogonal projection onto either $\mathbf{X}_0$ or $\mathbf{X}_m$, generated by $w_1, \ldots, w_k$. The equations (35) and (36) can be transformed into a system of integro- differential equations in the variables $g_{jn}(t)$. This system can be simplified further into an autonomous first-order system of differential equations, as shown below:

For $k = 1, 2, 3, \ldots, n$ we have that

$$\begin{aligned} (v_t^n, w_k) &= \left(\sum_{j=1}^{n} g'_{jn}(t) w_j, w_k\right), \\ &= \sum_{j=1}^{n} g'_{jn}(t)(w_j, w_k), \\ &= g'_{jn}(t). \end{aligned} \tag{37}$$

Similarly

$$
\begin{aligned}
(v^n \cdot \nabla)v^n, w_k) &= \sum_{j=1}^{n}((v^n \cdot \nabla)v^n w_j, w_k)g_{jn}(t), \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}((w_i \cdot \nabla)w_i, w_k)g_{in}(t)g_{jn}(t), \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} C_{ij}^k g_{in}(t)g_{jn}(t).
\end{aligned} \tag{38}
$$

Where $C_{ij}^k = ((w_i \cdot \nabla)w_i, w_k)$.

When looking at equation (34), we can determine the density using the following expression:

$$
\begin{aligned}
\rho^n(x,t) &= \rho_0(x) + \int_0^t v^n(x,s)e_3 \ \mathrm{d}s, \\
&= \rho_0(x) + \int_0^t \sum_{j=1}^{n} w_j(x)e_3 g_{jn}(s) \ \mathrm{d}s.
\end{aligned} \tag{39}
$$

Therefore, it can be inferred that.

$$
(\rho^n(x,t)e_3, w_k) = \langle \rho_0(x)e_3, w_k\rangle + \int_0^t \sum_{j=1}^{n}\langle (w_j(x)e_3)e_3, w_k\rangle g_{jn}(s) \ \mathrm{d}s.
$$

We substitute (37), (38) (39) in equation (36), and take $X(t)$ such that

$$
X(t) = (g_{1n}(t), g_{2n}(t), \ldots\ldots.., g_{nn}(t)),
$$

the system (36) is equivalent to

$$
X'(t) = F(x,t) + \int_0^t G(x,s) \ \mathrm{d}s, \tag{40}
$$

where

$$
F(x) = \begin{pmatrix} x^T M_1 X \\ x^T M_2 X \\ \vdots \\ x^T M_n X \end{pmatrix} + \begin{pmatrix} \langle \rho_0(x)e_3, w_1\rangle \\ \langle \rho_0(x)e_3, w_2\rangle \\ \vdots \\ \langle \rho_0(x)e_3, w_n\rangle \end{pmatrix},
$$

with

$$
M_k = \begin{pmatrix} C_{11}^k & C_{12}^k & \ldots & C_{1n}^k \\ C_{21}^k & C_{22}^k & \ldots & C_{2n}^k \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1}^k & C_{n2}^k & \ldots & C_{nn}^k \end{pmatrix},
$$

and

$$
G = \begin{pmatrix} \langle w_1(x)\cdot e_3, w_1\rangle & \langle w_2(x)\cdot e_3, w_1\rangle & \ldots & \langle w_n(x)\cdot e_3, w_1\rangle \\ \langle w_1(x)\cdot e_3, w_2\rangle & \langle w_2(x)\cdot e_3, w_2\rangle & \ldots & \langle w_n(x)\cdot e_3, w_2\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle w_1(x)\cdot e_3, w_n\rangle & \langle w_2(x)\cdot e_3, w_n\rangle & \ldots & \langle w_n(x)\cdot e_3, w_n\rangle \end{pmatrix}.
$$

Differentiating equation (40) and noting that

$$(x^T M_k X)' = (M_k x \cdot X)' = M_k x' \cdot X + M_k x \cdot X',$$

we get that

$$X''(t) = K(x, x'),$$

where

$$K_j(x, x') = M_j x' \cdot X + M_j x \cdot X' + (G \cdot X)_j, \text{ for } j = 1, 2, ..., n. \tag{41}$$

We introduce the notation $x' = z$, the equation (41) can be written as

$$\begin{pmatrix} x \\ z \end{pmatrix}' = \begin{pmatrix} z \\ K(x, z) \end{pmatrix} = A(x, z). \tag{42}$$

Since $A(x, z)$ is an infinitely differentiable vector field, then by the theory of ordinary differential equations, we conclude that (42) admits a maximal solution in an interval $[0, T_n]$.

Now, it is necessary to show that the times T_n do not depend on n and that the solutions are uniformly bounded in $\mathbf{H}^m$ to obtain a local solution of the equation using the limit step by a compactness argument.

The following estimation on v^n shows that $T_n = T$ is independent of n. We start with the proof of the following result.

Lemma 4.2.
For all $n \in \mathbb{N}$, there exists $T_* > 0$ such that

$$\sup_{0 \le t \le T_n} \| v^n(x, t) \|^2 \le \| v_0(x) \|^2 + \frac{g^2}{N^2} \| \rho_0(x) \|^2 . \tag{43}$$

Proof.
If we multiply by $g_{jn}(t)$ in the equation (36) and add in k, we get that

$$\left(\frac{\mathrm{d}}{\mathrm{dt}}\right) \langle v^n, w_k \rangle + \langle (v^n \cdot \nabla) v^n, w_k \rangle = \langle -g\rho^n e_3, w_k \rangle$$

$$\left(\frac{\mathrm{d}}{\mathrm{dt}}\right) \langle v^n, w_k g_{jn} \rangle + \langle (v^n \cdot \nabla) v^n, w_k g_{jn} \rangle = \langle -g\rho^n e_3, w_k g_{jn} \rangle,$$

so, we have

$$\langle v_t^n, v^n \rangle + \langle (v^n \cdot \nabla) v^n, v^n \rangle = \langle -g\rho^n e_3, v^n \rangle. \tag{44}$$

Note that we can apply the Stokes formula to the second term of the equation (44),

$$\begin{aligned} \langle (v^n \cdot \nabla) v^n, v^n \rangle &= \int_\Omega \sum_{i=1}^{3} v_i^n \partial_i v^n \cdot v^n \\ &= \sum_{i=1}^{3} \int_\Omega v_i^n \partial_i \left(\frac{|v^n|^2}{2} \right) \\ &= \int_\Omega \sum_{i=1}^{3} \partial_i v_i^n \frac{|v^n|^2}{2} \\ &= 0, \end{aligned}$$

that is,

$$\langle (v^n \cdot \nabla) v^n, v^n \rangle = 0.$$

Therefore,

$$\langle v_t^n, v^n \rangle = \langle -g\rho^n e_3, v^n \rangle.$$

On the other hand, considering the following proof equation will help us find a bound for v^n as follows

$$\begin{aligned} E(t) &= \frac{1}{2}\|v^n\|^2 + \frac{c}{2}\|\rho^n\|^2, \\ E(t) &= \frac{1}{2}\langle v^n, v^n \rangle + \frac{c}{2}\langle \rho^n, \rho^n \rangle. \end{aligned}$$

Thus, if we differentiate the above equation, it follows

$$\begin{aligned} E'(t) = \langle v_t^n, v^n \rangle + c\langle \rho_t^n, \rho^n \rangle \Longrightarrow E'(t) &= \langle -g\rho^n, v^n \rangle + c\left\langle \frac{N^2}{g} v^n, \rho^n \right\rangle, \\ &= \langle -g\rho^n, v^n \rangle + c\frac{N^2}{g}\langle v^n, \rho^n \rangle \text{ for } c = \frac{g^2}{N^2}, \\ &= g\langle -\rho^n, v^n \rangle + g\langle v^n, \rho^n \rangle, \\ &= 0. \end{aligned} \tag{45}$$

Therefore, $E'(t) = 0$, that is, $E(t)$ is constant, even more $E(t) = E(0)$. Then

$$\frac{1}{2}\|v^n\|^2 + \ c\|\rho^n\|^2 = \|v(x,0)\|^2 + \ c_0\|\rho(x,0)\|^2, \tag{46}$$

where $c_0 = \dfrac{g^2}{N^2}$. This way, we have

$$\sum_{i=1}^{n} \mid g_{in}(t) \mid^2 = \| \ v^n \ \|^2 = \| \ X(t) \ \|^2,$$

which tells us that $\| \ X(t) \ \|^2$ will not explode in a finite time, as we wanted to show. □

Thus, from this lemma, we can conclude that for n fixed, the function $v^n(x,t)$ is uniformly bounded in $\mathbf{L}^2((0,T);\mathbf{L}^2(\Omega)^3)$ independently of n, which implies that it will not explode in a finite time.

On the other hand, we multiply the equation (36) by $\lambda_k g_k$ and add in k for $k = 1, 2, ...n$, and we get

$$\begin{aligned} \langle v_t^n, w_k g_k \rangle \lambda_k &= \langle -g\rho^n e_3, w_k g_k \rangle \lambda_k - \langle (v^n \cdot \nabla) v^n, w_k g_k \rangle \lambda_k, \\ \langle v_t^n, v^n \rangle_m &= \langle -g\rho^n, v^n \rangle_m - \langle (v^n \cdot \nabla) v^n, v^n \rangle_m, \\ \frac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{dt}}\right)\|v^n\|_m^2 &= \langle (-g\rho^n - (v^n \cdot \nabla) v^n), v^n \rangle_m. \end{aligned} \tag{47}$$

Now, applying Cauchy's Inequality, we obtain bounds for each one of the terms on the right-hand side of the equation (47), that is,

$$\begin{aligned} |\langle -g\rho^n, v_n \rangle_m| &\le \|g\rho^n\|_m \|v\|_m, \\ |\langle v^n \cdot \nabla \rangle v^n, v^n)_m| &\le c\|v\|_m^3, \end{aligned}$$

which when replacing them in the equation (47), we have

$$\frac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{dt}}\right)\|v^n\|_m^2 \le c_1\|v^n\|_m^2 + c_2\|g\rho^n\|_m,$$

where the term ρ^n is bounded by

$$\begin{aligned}
\rho^n(x,t) &= \rho_0(x) + c\int_0^t v_3^n(x,s)\mathrm{d}s,\\
\|\rho^n\|_m &\le \|\rho_0\|_m + c\int_0^t \|v_3^n\|_m \mathrm{d}s,\\
&\le \|\rho_0\|_m + c\int_0^t \|v_n\|_m \mathrm{d}s,\\
&\le \|\rho_0\|_m + \int_0^t \left(\frac{1}{2}c^2 + \frac{1}{2}\|v_n\|^2\right)\mathrm{d}s,\\
&\le \|\rho_0\|_m + \frac{c}{2}t + \int_0^t \|v_n\|^2 \mathrm{d}s\\
&\le \|\rho_0\|_m + \frac{c^2}{2}t + \frac{1}{2}\int_0^t \|v_n\|_m^2 \mathrm{d}s,
\end{aligned}$$

then, we have

$$\left(\frac{\mathrm{d}}{\mathrm{dt}}\right)\|v^n\|_m^2 \le c_1\|v^n\|_m^2 + c_2\|\rho_0\|_m + c_3 t + \int_0^t \|v^n\|_m^2 \ \mathrm{d}s.$$

Thus, it follows that $\|v^n(t)\|_m \le g(t)$ for all $t < \inf(T, T_0)$, where $g(t)$ is the solution of the differential equation

$$\left(\frac{\mathrm{d}}{\mathrm{dt}}\right) g(t) = c_1(g(t))^2 + c_2\|\rho_0\|_m + c_3 t + \int_0^t (g(s))^2 \ \mathrm{d}s.$$

Therefore, considering (46), it follows that v^n is bounded in

$$L^\infty((0,T_*);\mathbf{L}^2(\Omega)^3) \cap \mathbf{L}^2((0,T_*);\mathbf{H}^m(\Omega)^3) \text{ for all } T_* > 0. \tag{48}$$

In this way, we get the following lemma

Lemma 4.3.
For all $n \in \mathbb{N}$ and for all $t \in [0, T_*]$ there exists $C > 0$ independent of v^n such that

$$\int_0^T \| \ v^n(x,\tau) \ \|_m^2 \ \ \mathrm{d}\tau \le C(\| \ v_0(x) \ \|, \| \ \rho_0(x) \ \|, T_*). \tag{49}$$

Now, we need an estimate for v_t^n; for this, let's see the following result.

Lemma 4.4. *For all $n \in \mathbb{N}$, there exists $T_* > 0$ such that*

$$v_t^n \ \textit{is bounded in} \ \mathbf{L}^\infty((0,T_*);\mathbf{H}^m(\Omega)^3) \ \textit{for} \ \ \textit{all} \ T_* > 0. \tag{50}$$

Proof.
We start from the equation given by (36)

$$\langle v_t^n, w_k\rangle + \langle (v^n\cdot\nabla)v^n, w_k\rangle = \langle -g\rho^n e_3, w_k\rangle,$$

Then, we take its derivative to get the following expression

$$\langle v_{tt}^n, w_k\rangle + \langle (v_t^n\cdot\nabla)v^n, w_k\rangle + \langle (v^n\cdot\nabla)v_t^n, w_k\rangle = \langle -g\rho_t^n e_3, w_k\rangle,$$

which we can also write in terms of P, that is,

$$\langle v_{tt}^n, w_k\rangle + \langle P(v_t^n\cdot\nabla)v^n, w_k\rangle + \langle P(v^n\cdot\nabla)v_t^n, w_k\rangle = \langle -Pg\rho_t^n e_3, w_k\rangle,$$

multiplying by $g'_{jn}\lambda_k$ the previous equation, we get the following expression

$$\begin{aligned}(v_{tt}^n, g'_{jn}w_k)\lambda_k + \langle P(v_t^n\cdot\nabla)v^n, g'_{jn}w_k\rangle\lambda_k + \langle P(v^n\cdot\nabla)v_t^n, g'_{jn}w_k\rangle\lambda_k \\ = (-Pg\rho_t^n e_3, g'_{jn}w_k)\lambda_k,\end{aligned}$$

by adding in k and applying (48) we get

$$\langle v_{tt}^n, v_t^n\rangle_m + \langle (v_t^n\cdot\nabla)v^n, v_t^n\rangle_m + \langle (v^n\cdot\nabla)v_t^n, v_t^n\rangle_m = \langle -g\rho_t^n e_3, v_t^n\rangle_m,$$

On the other hand, as

$$\langle (v^n\cdot\nabla)v_t^n, v_t^n\rangle = 0 \quad\text{and}\quad \rho_t^n(x,t) = \frac{N^2}{g}v_3^n\cdot e_3,$$

then

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}||v_t^n||^2 = -\langle (v_t^n\cdot\nabla)v^n, v_t^n\rangle_m - N^2(v_3^n e_3, v_t^n)_m.$$

Now, we will take each one of the terms on the right side of the previous equation separately, and we will find bounds for each of them:

Indeed:

$$\begin{aligned}|N^2\langle v_3^n e_3, v_t^n\rangle_m| &\le N^2\|v_3^n e_3\|_m\|v_t^n\|_m,\\ &\le N^2\|v^n\|_m\|v_t^n\|_m,\\ &\le \frac{N^2}{2}(\|v^n\|_m^2 + \|v_t^n\|_m^2),\\ &\le \frac{N^2}{2}(\|v(x,0)\|_m^2 + \ c_0\|\rho(x,0)\|_m^2) + \frac{N^2}{2}\|v_t^n\|_m^2,\end{aligned}$$

therefore

$$|N^2\langle v_3^n e_3, v_t^n\rangle_m| \le \frac{N^2}{2}(\|v(x,0)\|_m^2 + \ c_0\|\rho(x,0)\|_m^2) + \frac{N^2}{2}\|v_t^n\|_m^2.$$

Using HÃ¶lder's inequality generalized, we obtain

$$\begin{aligned}| < (v_t^n\cdot\nabla)v^n, v_t^n >_m | &\le \|v_t^n\|_m\|\nabla\cdot v^n\|_m\|v_t^n\|_m,\\ &= C\|v_t^n\|^2\|v^n\|_m.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}||v_t^n||^2 &\le C\|v_t^n\|_m^2\|v^n\|_m + \frac{N^2}{2}(\|v(x,0)\|_m^2 + \ c_0\|\rho(x,0)\|_m^2) + \frac{N^2}{2}\|v_t^n\|_m^2,\\ &\le \left(\|v^n\|_m + \frac{N^2}{2}\right)\|v_t^n\|_m^2 + \frac{N^2}{2}(\|v(x,0)\|_m^2 + c_0\|\rho(x,0)\|_m^2),\end{aligned}$$

Now, using equation (48), we have that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}||v_t^n||_m^2 \le a\|v_t^n\|_m^2 + b,$$

where $a = C + \dfrac{N^2}{2}$ and $b = \dfrac{N^2}{2}(\|v(x,0)\|_m^2 + \ c_0\|\rho(x,0)\|_m^2)$. So, we get

$$\frac{1}{2}\int_0^t \left(\frac{\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}||v_t^n||_m^2}{a\|v_t^n\|_m^2 + b} \right) \leq t,$$

note that if $u(t) = \|v_t^n\|_m^2$, then

$$\int_{u(0)}^{u(t)} \left(\frac{du}{au+b} \right) \leq t \Longrightarrow \frac{1}{a}\ln|au+b|\Bigg|_{u(0)}^{u(t)} \leq t.$$

Thus,

$$\begin{aligned} u(t) &\leq \frac{1}{a}e^{at+\ln|au(0)+b|} - b, \\ \|v_t^n\|_m^2 &\leq \frac{1}{a}e^{at+\ln|\|v_t^n(0)\|_m^2+b|} - \frac{b}{a}, \\ \|v_t^n\|_m^2 &\leq \frac{1}{a}e^{at+\ln|\|v_t^n(0)\|_m^2+b|} - \frac{b}{a}. \end{aligned}$$

Therefore, it follows that v_t^n in $\mathbf{L}^\infty((0,T_*);\mathbf{H}^m(\Omega)^3)$, as wanted to show. □

Now, taking into account equations (48) and (50) and applying the Rellich-Kondrachov Theorem given by (2.3), we can find a subsequence v^n that strongly converges in $\mathbf{L}_2((0,T_*)\times\Omega)$, which corresponds to the following lemma

Lemma 4.5.

$$v^n \to v \in \mathbf{L}_2((0,T_*)\times\Omega) \tag{51}$$

and

$$\partial_t v^n \to \partial_t v \in \mathbf{L}_2((0,T_*)\times\Omega). \tag{52}$$

Since v^n is bounded on $\mathbf{H}^m$, then for $|\alpha| \leq m$, we have that $\partial_\alpha v^n$ is bounded and weakly converges in $\mathbf{L}^2(\Omega)$. By uniqueness of limits we deduce that $v(t) \in \mathbf{H}^m$ and $\partial_\alpha v^n(t)$ weakly converges in $\mathbf{L}^2(\Omega)$ for all $\partial_\alpha v(t)$ and $t \leq T_*$.

Note that $|v(t)|_{\mathbf{H}^m} \leq \liminf |v^n(t)|_{\mathbf{H}^m}$. Therefore, we have

$$v \in L^\infty((0,T_*),\mathbf{H}^m(\Omega)).$$

5. Existence and Uniqueness of Solutions

Considering the estimations of the previous section, we will state the following existence theorem.

Theorem 5.1. *Let Ω be a bounded domain of $\mathbb{R}^3$, with smooth boundary, and let $v_0(x) \in \mathbf{H}^m(\Omega)$, then there exists an interval $[0,T_*]$ and functions $v(x,t), p(x,t)$ and $\rho(x,t)$ that satisfy system (24) in $Q_{T_*} = \Omega\times[0,T_*]$ in the sense (25), and also satisfy the following properties:*

$$\begin{cases} (v,p) \in (\mathbf{L}^\infty((0,T_*),\mathbf{X}_0) \cap \mathbf{L}^2((0,T_*)),\mathbf{H}^m(\Omega))) \times \mathbf{L}^2((0,T_*)),\mathbf{H}^{m+1}(\Omega)^3), \\ \rho \in \mathbf{L}^\infty((0,T_*),\mathbf{L}^2(\Omega)), \\ \dfrac{\partial v}{\partial t} \in \mathbf{L}^2((0,T_*),\mathbf{X_0}(\Omega)). \end{cases} \tag{53}$$

Proof.
With these hypotheses we have all the estimates given in (46), (50), (51) and (52), thus obtaining a subsequence of $\{v^n\}$, which we continue to call $\{v^n\}$ and a function v, which satisfies that

$$\begin{cases} v^n \rightharpoonup v \text{ weak-* in } \mathbf{L}^\infty((0,T_*),\mathbf{X}_0(\Omega)), \\ \dfrac{\partial v^n}{\partial t} \rightharpoonup \dfrac{\partial v}{\partial t} \text{ weakly in } \mathbf{L}^2((0,T_*),\mathbf{X}_0(\Omega)), \\ v^n \rightharpoonup v \text{ weakly in } \mathbf{L}^2((0,T_*)\cap \mathbf{H}^m(\Omega)^3), \\ v^n \to v \text{ strong in } \mathbf{L}^2((0,T_*),\mathbf{X}_0(\Omega)). \end{cases} \tag{54}$$

With these convergences, it is easy to see that v satisfies the regularity properties of the theorem. Since in Banach spaces, the norms are semicontinuous functions, we have that

- $\| v(\cdot,t) \|_{\mathbf{X_0}(\Omega)} \le \liminf_{n\to\infty} \|v^n(\cdot,t)\|_{\mathbf{X_0}(\Omega)} \le C_1$, then

$$v \in \mathbf{L}^\infty((0,T_*);\mathbf{X}_0(\Omega)).$$

- $\| v(\cdot,t) \|_{\mathbf{H}^m(\Omega)} \le \liminf_{n\to\infty} \|v^n(\cdot,t)\|_{\mathbf{H}^m(\Omega)} \le C_2$, then

$$v \in \mathbf{L}^2((0,T_*);\mathbf{H}^m(\Omega)).$$

- $\left\|\dfrac{\partial v}{\partial t}(\cdot,t)\right\|_{\mathbf{X}_0(\Omega)} \le \liminf_{n\to\infty} \| v_t^n(\cdot,t) \|_{\mathbf{X}_0(\Omega)} \le C_3$, then

$$v_t \in \mathbf{L}^2((0,T_*),\mathbf{X}_0(\Omega)).$$

- $\| v(\cdot,t) \|_{\mathbf{H}^m(\Omega)^3} \le \liminf_{n\to\infty} \|v^n(\cdot,t)\|_{\mathbf{H}^m(\Omega)^3} \le C_4$, then

$$v \in \mathbf{L}^2((0,T_*);\mathbf{H}^m(\Omega)^3).$$

On the other hand, let us show that v satisfies (24). For this, let us see that v^n and ρ^n satisfy (24). Indeed: Let

$$\phi = \sum_{k=1}^{m} H_k(t)w_k,\ H_k(t) \in C^1([0,T_*]), H_k(T_*) = 0. \tag{55}$$

If we multiply (35) by $H_k(t)$ and add in k from 1 to m, we obtain that

$$\langle v_t^n + (v^n\cdot\nabla)v^n + g\rho^n e_3, \phi\rangle = 0.$$

Now, integrating from 0 to T_*, and using integration by parts it follows that

$$\int_0^T \{\langle v^n,\phi_t\rangle + \langle v^n,(v^n\cdot\nabla)\phi\rangle + g\langle \rho^n e_3,\phi\rangle\}\ \mathrm{d}t + \langle v^n(0),\phi(0)\rangle = 0. \tag{56}$$

The weak convergence of ρ^n in $\mathbf{L}^2((0,T_*);\mathbf{L}^2(\Omega))$, implies that

$$\int_0^{T_*} \langle \rho^n e_3,\phi\rangle\ \mathrm{d}t \ \overrightarrow{n\to\infty}\ \int_0^{T_*} \langle \rho e_3,\phi\rangle\ \mathrm{d}t.$$

In this way, taking limit as $n \longrightarrow \infty$ in (56), we obtain (24), for all ϕ of the form (55), provided that

$$\int_0^{T_*} \langle v^n, (v^n \cdot \nabla)\phi \rangle \ \mathrm{dt} \xrightarrow[n \to \infty]{} \int_0^{T_*} \langle v, (v \cdot \nabla)\phi \rangle \ \mathrm{dt}.$$

This is evidenced considering v^n is uniformly bounded in $\mathbf{L}^\infty((0, T_*); \mathbf{X}_0(\Omega))$; $v^n \longrightarrow v$ strongly in $\mathbf{L}^2(Q_{T_*})$ and integrating from 0 to T_* the identity

$$\langle v^n, (v^n \cdot \nabla)\phi \rangle - \langle v, (v \cdot \nabla)\phi \rangle = \langle v^n - v, (v \cdot \nabla)\phi \rangle + \langle v^n, ((v^n - v) \cdot \nabla)\phi \rangle.$$

Thus, taking a limit when $n \to \infty$ in (56), we obtain that

$$\int_0^{T} \{\langle v, \phi_t \rangle + \langle v, (v \cdot \nabla)\phi \rangle + g\langle \rho e_3, \phi \rangle\} \ \mathrm{d}t + \langle v(0), \phi(0) \rangle \ \mathrm{d}t = 0. \tag{57}$$

Due to the density of the set (55), we have the identity (57). □

Theorem 5.2. *Uniqueness of the solutions.*

Proof.
To demonstrate that the solutions found for the system (21) in its vector form are unique, we define two solutions denoted by $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$. With scalar pressure fields p_1, p_2 and densities ρ_1 and ρ_2 respectively, which satisfy the conditions of the system (21) with the same initial data. Then, we take the difference of these two solutions, $U = w - v$ and $P = p_2 - p_1$, according to the system (21), it is verified that

$$\frac{\partial U}{\partial t} = -\nabla P + \begin{pmatrix} v \cdot \nabla v_1 - w \cdot \nabla w_1 \\ v \cdot \nabla v_2 - w \cdot \nabla w_2 \\ v \cdot \nabla v_3 - w \cdot \nabla w_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -\frac{\sqrt{g}}{N}(\rho_2 - \rho_1) \cdot e_3 \end{pmatrix}. \tag{58}$$

From the equation (22) - (23), we also have that

$$\rho_2(x,t) - \rho_1(x,t) = \frac{\sqrt{g}}{N} \int_0^t (w_3(x,t) - v_3(x,t)) \mathrm{d}s.$$

Adding and subtracting the term $(v \cdot \nabla)w$ it is verified that

$$\frac{\partial U}{\partial t} = -\nabla P + \begin{pmatrix} v \cdot \nabla U_1 - U \cdot \nabla w_1 \\ v \cdot \nabla U_2 - U \cdot \nabla w_2 \\ v \cdot \nabla U_3 - U \cdot \nabla w_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -\frac{\sqrt{g}}{N}(\rho_2 - \rho_1) \cdot e_3 \end{pmatrix}. \tag{59}$$

Now, multiplying by U, integrating by parts, and using the fact that
$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|U\|^2 = \langle \frac{d}{dt}U, U \rangle$ and $\langle \nabla P, U \rangle = 0$ and $\langle (v\nabla)U, U \rangle = 0$ to define (58), we get

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|U\|^2 = &\left\langle \begin{pmatrix} V \cdot \nabla v_1 - W \cdot \nabla w_1 \\ V \cdot \nabla v_2 - W \cdot \nabla w_2 \\ V \cdot \nabla v_3 - W \cdot \nabla w_3 \end{pmatrix}, U \right\rangle + \\ &+ \frac{g}{N}^2 \left\langle \begin{pmatrix} 0 \\ 0 \\ \int_0^t (w_3(x,t) - v_3(x,t)) \ \mathrm{d}s \end{pmatrix}, U \right\rangle. \end{aligned} \tag{60}$$

On the other hand, we have the following expression

$$(V \cdot \nabla V) - (W \cdot \nabla)W = -(U \cdot \nabla)W - (V \cdot \nabla)U,$$

with $\langle (V\cdot\nabla)U, U\rangle = 0$. So, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|U\|^2 = -\frac{g}{N}^2\left\langle \int_0^t (w_3(x,t)-v_3(x,t))\ \mathrm{d}s, w_3 - v_3\right\rangle - \langle (U\cdot\nabla)w, U\rangle\,. \tag{61}$$

In the second term of (61), we can be bound by taking into account the triangular inequality, that is,

$$|\langle (U\cdot\nabla)W, U\rangle| \le \|U\|^2\|W\|, \text{ in this case } U\in L^4 \text{ and } W\in W^{1,2}.$$

For the first term of (61), we have the following estimate:

$$\begin{aligned}
&\left|\left\langle \int_0^t (w_3(x,t)-v_3(x,t))\ \mathrm{d}s, w_3-v_3\right\rangle\right| \\
&\le \int_\Omega \left|\int_0^t (w_3(x,s)-v_3(x,s))\ \mathrm{d}s\right|\ |w_3(x,t)-v_3(x,t)|\ \mathrm{d}x, \\
&\le \frac{1}{2}\int_\Omega \left(\int_0^t (w_3(x,s)-v_3(x,s)\ \mathrm{d}s\right)^2\ \mathrm{d}x + \frac{1}{2}\int_\Omega (w_3(x,t)-v_3(x,t))^2\ \mathrm{d}x, \\
&\le \frac{1}{2}t\int_\Omega\int_0^t (w_3(x,s)-v_3(x,s))^2\ \mathrm{d}s\ \mathrm{d}x + \frac{1}{2}\int_\Omega (w_3(x,t)-v_3(x,t))^2\ \mathrm{d}x, \\
&\le \frac{1}{2}t\int_0^t\int_\Omega (w_3(x,s)-v_3(x,s))^2\ \mathrm{d}x\ \mathrm{d}s + \frac{1}{2}\int_\Omega (w_3(x,t)-v_3(x,t))^2\ \mathrm{d}x,
\end{aligned}$$

this is

$$\left|\left\langle \int_0^t (w_3(x,t)-v_3(x,t))\ \mathrm{d}s, w_3-v_3\right\rangle\right| \le \frac{1}{2}t\int_0^t \|U\|^2\ \mathrm{d}s + \frac{1}{2}\|U\|^2,$$

in this way,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|U\|^2 \le \frac{1}{2}t\int_0^t \|U\|^2\ \mathrm{d}s + \frac{1}{2}\|U\|^2 + \|U\|^2\|W\|. \tag{62}$$

Now, we consider the initial value problem

$$\begin{cases} y' = f(t,y) = \phi(t)y + t\displaystyle\int_0^t y(s)\mathrm{d}s, \\ \qquad y(0) = 0, \end{cases} \tag{63}$$

where the solution of the problem (63) is given by $y=0$. Then, using the comparison principle, any solution of the differential inequality

$$v' \le f(t,v),$$

with $v(0)\le 0$, which satisfies that $v(t)\leqslant 0$. Thus, applying (62) it follows that $\|U\|^2 = 0$, which implies that $V = W$. □

6. Conclusion

In this paper, we demonstrate the existence and unique solution of a nonlinear system of partial differential equations related to stratified fluids within a bounded domain in three dimensions. We open the door to future studies of more general nonlinear systems, considering stratification, heat transfer, and salinity, as well as exploring some optimization problems whose solutions are constraints to the systems discussed in our article, following the philosophy of [5]. We could also extend our results to environments that involve fractional derivatives like in [2], [3], [6] and [21].

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