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# Resolvents of the equilibrium problem with generalized perturbations on complete geodesic spaces

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**Abstract**

In this paper, we study a class of resolvent operators for the equilibrium problem on a complete geodesic space. We prove such an operator defined by using a strictly midpoint convex perturbation function is well-defined as a single-valued mapping. We also show its natures.

*Keywords:* equilibrium problem resolvent CAT($\kappa$) space geodesic space nonspreading mapping.
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## 1. Introduction

In 1994, Blum and Oettli [1] proposed a notion of an equilibrium problem. It is a problem to find a point $z \in X$ such that $f(z, y) \geq 0$ for any $y \in X$ for a nonempty set $X$ and given $f\colon X^2 \to \mathbb{R}$. This problem is one of the generalization of the convex minimization problem. Indeed, putting $f(z, y) := g(y) - g(z)$ for $g\colon X \to \mathbb{R}$ and $z, y \in X$, we obtain $f(z, y) \geq 0$ for any $y \in X$ if and only if $g(z) = \min g$. The equilibrium problem also includes various nonlinear problems, such as fixed point problems, minimax problems, Nash equilibria, and so on.

In this paper, we consider equilibrium problems on geodesic spaces. The class of geodesic spaces includes nonlinear infinite-dimensional spaces in general, and we know that resolvent operator plays an important role to solve an equilibrium problem on geodesic spaces. In 2005, Combettes and Hirstoaga [3] showed that

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an resolvent operator $R_f\colon \mathcal{H} \to K$ is well-defined as a single-valued mapping on a Hilbert space $\mathcal{H}$ and its nonempty closed convex subset $K$. That operator $R_f$ is defined by

$$R_f x = \left\{ z \in K \;\middle|\; \inf_{y \in K} \left( f(z, y) + \langle z - x, y - z \rangle \right) \geq 0 \right\}$$

for each $x \in H$ and given $f\colon K^2 \to \mathbb{R}$. Particularly noteworthy is that the set of all fixed points of $R_f$ coincides with the set of all solutions to the equilibrium problem for $f$. Therefore, we can use the techniques of the fixed point theory to solve the equilibrium problem. Later, some researchers showed that resolvent operators for the equilibrium problem can be defined in complete CAT(0) spaces [7], complete CAT(1) spaces [6], and complete CAT($-1$) spaces [11]. These resolvents were defined by the following forms:

- In a complete CAT(0) space with CHFP (Kimura and Kishi [7]):

$$R_f x = \left\{ z \in K \;\middle|\; \inf_{y \in K} \left( f(z, y) + \frac{1}{2} d(x, y)^2 - \frac{1}{2} d(x, z)^2 \right) \geq 0 \right\};$$

- in an admissible complete CAT(1) space with CHFP (Kimura [6]):

$$R_f x = \left\{ z \in K \;\middle|\; \inf_{y \in K} \left( f(z, y) - \log \cos d(x, y) + \log \cos d(x, z) \right) \geq 0 \right\};$$

- in a complete CAT($-1$) space with CHFP (Kimura and Ogihara [11]):

$$R_f x = \left\{ z \in K \;\middle|\; \inf_{y \in K} \left( f(z, y) + \cosh d(x, y) - \cosh d(x, z) \right) \geq 0 \right\}.$$

Recently, in 2023, the authors shows that the resolvent for the equilibrium problem can be defined in general complete CAT($\kappa$) spaces with the convex hull finite property (CHFP) as follows:

**Theorem 1.1** ([9]). *Let $X$ be an admissible complete* CAT($\kappa$) *space. Suppose that $X$ has the convex hull finite property. Let $K$ be a nonempty closed convex subset of $X$ and $f$ a real function on $K^2$ with conditions* (E1)–(E4). *Let $\varphi\colon [0, c_\kappa(D_\kappa/2)[ \to [0, \infty[$ be a strictly increasing and differentiable function such that $\varphi'$ is continuous and nondecreasing. Define $\Phi\colon [0, D_\kappa/2[ \to [0, \infty[$ by $\Phi d := \Phi(d) = \varphi(c_\kappa(d))$ for $d \in [0, D_\kappa/2[$. In addition, assume the following:*

- *If $\kappa \leq 0$ and $K$ is unbounded, then assume*

$$\liminf_{\substack{d(v,z) \to \infty \\ z \in K}} \frac{f(v, z)}{d(v, z)} + \liminf_{d \to \infty} \frac{\Phi d}{d} > 0$$

  *for all $v \in K$;*

- *if $\kappa > 0$, then assume $\lim_{d \to D_\kappa/2} \Phi d = \infty$;*

- *otherwise, give no additional assumptions.*

*Define a subset $R_f x$ of $K$ by*

$$R_f x = \left\{ z \in K \;\middle|\; \inf_{y \in K} \left( f(z, y) + \Phi d(x, y) - \Phi d(x, z) \right) \geq 0 \right\}$$

*for each $x \in X$. Then the following hold:*

(i) *$R_f x$ consists of one point for every $x \in X$, and thus $R_f\colon X \to K$ is defined as a single-valued mapping;*

(ii) *$R_f$ satisfies the following inequality for any $x_1, x_2 \in X$:*

$$\begin{aligned}&\left(\psi(D_1)c''_\kappa(D_1) + \psi(D_2)c''_\kappa(D_2)\right) c_\kappa(d(R_f x_1, R_f x_2))\\ &\qquad \le \psi(D_1)\left(c_\kappa(d(x_1, R_f x_2)) - c_\kappa(D_1)\right) + \psi(D_2)\left(c_\kappa(d(x_2, R_f x_1)) - c_\kappa(D_2)\right),\end{aligned}$$

*where $\psi := \varphi' \circ c_\kappa$, $D_1 = d(x_1, R_f x_1)$, and $D_2 = d(x_2, R_f x_2)$;*

(iii) *$R_f$ is firmly vicinal with $\psi$;*

(iv) *$R_f$ is $\Delta$-demiclosed; and if $F(R_f) \neq \emptyset$, then $R_f$ is quasinonexpansive and asymptotically regular;*

(v) *The set $F(R_f)$ of all fixed points of $R_f$ coincides with* Equil $f$ *which is a set of all solutions to the equilibrium problem for $f$; thus the set* Equil $f$ *is closed and convex.*

This result contains the all results about the properties of resolvents in [7], [6], and [11]. In fact, these are the case of $(\kappa, \varphi(t)) = (0, t)$, $(1, -\log(1-t))$, $(-1, t+1)$, respectively.

Incidentally, in CAT($\kappa$) spaces, the concept of a resolvent operator for the convex minimization problem has also been proposed. In 1998, Mayer [13] proposed a single-valued operator $J_f \colon X \to X$ for a proper lower semicontinuous convex function $f \colon X \to \,]-\infty, \infty]$ on a complete CAT(0) space $X$ by

$$J_f x = \operatorname*{argmin}_{y \in X} \left\{ f(y) + \frac{1}{2} d(y, x)^2 \right\}$$

for all $x \in X$. Then the set of all fixed points of $R_f$ is identical to the set of all minimizers of $f$. Later, in 2016, Kimura and Kohsaka [8] defined a resolvent operator $J_f \colon X \to X$ on an admissible complete CAT(1) space $X$ by

$$J_f x = \operatorname*{argmin}_{y \in X} \left\{ f(y) + \tan d(y, x) \sin d(y, x) \right\}$$

for all $x \in X$. They showed that is well-defined as a single-valued operator. Moreover, in 2019, Kajimura and Kimura [4] show the following operator $J_f \colon X \to X$ is single-valued on a complete CAT($-1$) space $X$:

$$J_f x = \operatorname*{argmin}_{y \in X} \left\{ f(y) + \tanh d(y, x) \sinh d(y, x) \right\}$$

for all $x \in X$.

In an admissible complete CAT(1) space $X$ and its nonempty closed convex subset $K$, we can obtain that a resolvent $R_f \colon X \to K$ of the equilibrium problem for $f \colon K^2 \to \mathbb{R}$ defined by

$$R_f x = \left\{ z \in K \;\middle|\; \inf_{y \in K} \left( f(z, y) + \tan d(x, y) \sin d(x, y) - \tan d(x, z) \sin d(x, z) \right) \ge 0 \right\}$$

is well-defined as a single-valued mapping from Theorem 1.1, see [9]. However, in a complete CAT($-1$) space $X$ and similarly defined $K$, we cannot get a well-definedness of an operator $S_f \colon X \to K$ defined by

$$S_f x = \left\{ z \in K \;\middle|\; \inf_{y \in K} \left( f(z, y) + \tanh d(x, y) \sinh d(x, y) - \tanh d(x, z) \sinh d(x, z) \right) \ge 0 \right\}$$

from Theorem 1.1. Indeed, a perturbation function $\Phi d = \tanh d \sinh d$ is obtained by a case where $\varphi(t) = t + 1 - 1/(t+1)$ for $t \in [0, \infty[$, and then $\varphi'(t) = 1 + 1/(t+1)^2$ holds, which implies that $\varphi'$ is not nondecreasing.

In this paper, we study a class of resolvent operators including the operator $S_f$, and prove that such a mapping is well-defined as a single-valued mapping. We also show its natures.

## 2. Preliminaries

In this paper, $F(T)$ stands for the a set of all fixed points of a mapping $T$ from a set $X$ into itself.

Let $X$ be a metric space and $f$ a mapping from $X$ into $]-\infty,\infty]$. Let us denote an effective domain of $f$ by $\operatorname{dom}(f) = \{x \in X \mid f(x) \neq \infty\}$. A function $f$ is said to be *proper* if $\operatorname{dom}(f) \neq \emptyset$. $f$ is said to be *coercive* if $f(y) \to \infty$ whenever $d(x,y) \to \infty$ for some $x \in X$.

Let $(X,d)$ be a metric space. A mapping $T\colon X \to X$ is said to be *hyperbolically nonspreading* [4] if

$$2\cosh d(Tx,Ty) \leq \cosh d(x,Ty) + \cosh d(y,Tx)$$

holds for any $x,y \in X$. A mapping $T\colon X \to X$ is said to be *asymptotically regular* [4] if $\{d(T^{n+1}x, T^n x)\}$ converges to $0$ as $n \to \infty$ for every $x \in X$.

### 2.1. Geodesic spaces

Let $X$ be a metric space with metric $d$. For a two points $x,y \in X$, a mapping $\gamma_{x,y}\colon [0,1] \to X$ is called a *geodesic* joining $x$ and $y$ if $\gamma_{x,y}(0) = y$, $\gamma_{x,y}(1) = x$, and $d(\gamma_{x,y}(s), \gamma_{x,y}(t)) = |s-t|d(x,y)$ hold for any $s,t \in [0,1]$.

For $D \in ]0,\infty]$, a metric space $(X,d)$ is called a uniquely $D$-geodesic space if for each two points $x$ and $y$ with $d(x,y) < D$, there exists a unique geodesic joining $x$ and $y$. Let $x,y \in X$ arbitrarily and take a unique geodesic $\gamma_{x,y}$ joining these points. For any $t \in [0,1]$, put $tx \oplus (1-t)y = \gamma_{x,y}(t)$ and call it a *convex combination* of $x$ and $y$. It follows that $d(x, tx \oplus (1-t)y) = (1-t)d(x,y)$ and $d(y, tx \oplus (1-t)y) = td(x,y)$ immediately. Furthermore, put $[x,y] = [y,x] = \gamma_{x,y}([0,1]) = \{tx \oplus (1-t)y \mid t \in [0,1]\}$. It is often called a *geodesic segment* joining $x$ and $y$.

Let $X$ be a uniquely $D$-geodesic space. A subset $C$ of $X$ is said to be *convex* if $[x,y] \subset C$ holds for any $x,y \in X$.

Let $M_\kappa$ be a 2-dimensional model space with a metric $d'$ and a constant curvature $\kappa \in \mathbb{R}$ defined by

$$M_\kappa = \begin{cases} \dfrac{1}{\sqrt{\kappa}}\mathbb{S}^2 & (\text{if } \kappa > 0); \\ \mathbb{R}^2 & (\text{if } \kappa = 0); \\ \dfrac{1}{\sqrt{-\kappa}}\mathbb{H}^2 & (\text{if } \kappa < 0), \end{cases}$$

where $kM$ denotes a model space with a metric $d'$ defined by $d'(x',y') = kd(x,y)$ for a given model space $(M,d)$, a real number $k > 0$, two points $x,y \in M$, and its corresponding points $x',y' \in kM$. We write a diameter of $M_\kappa$ as $D_\kappa$, which is equal to

$$D_\kappa = \begin{cases} \infty & (\text{if } \kappa \leq 0); \\ \dfrac{\pi}{\sqrt{\kappa}} & (\text{if } \kappa > 0). \end{cases}$$

Then $M_\kappa$ is a complete uniquely $D_\kappa$-geodesic space.

Let $\kappa \in \mathbb{R}$ and $X$ a uniquely $D_\kappa$-geodesic space. Put $\triangle(x,y,z) = [x,y] \cup [y,z] \cup [z,x]$ for each $x,y,z \in X$ such that $d(x,y) + d(y,z) + d(z,x) < 2D_\kappa$ and call it a *geodesic triangle* with vertices $x$, $y$ and $z$. For each geodesic triangle $\triangle(x,y,z)$, there exists $\overline{x}, \overline{y}, \overline{z} \in M_\kappa$ such that $d(x,y) = d'(\overline{x},\overline{y})$, $d(y,z) = d'(\overline{y},\overline{z})$, and $d(z,x) = d'(\overline{z},\overline{x})$, since $d(x,y) + d(y,z) + d(z,x) < 2D_\kappa$. Then define a *comparison triangle* $\overline{\triangle}(\overline{x},\overline{y},\overline{z})$ of $\triangle(x,y,z)$ by $[\overline{x},\overline{y}] \cup [\overline{y},\overline{z}] \cup [\overline{z},\overline{x}]$. Take a point $p \in \triangle(x,y,z)$ arbitrarily. Then there exists a corresponding point $\overline{p} \in \overline{\triangle}(\overline{x},\overline{y},\overline{z})$ to $p$ such that the distances from two adjacent vertices are identical. We call that point $\overline{p}$ a *comparison point* of $p$.

For $\kappa \in \mathbb{R}$, let $(X,d)$ a uniquely $D_\kappa$-geodesic space and $(M_\kappa, d')$ a model space. We call $X$ a CAT($\kappa$) space if for any $\triangle := \triangle(x,y,z)$ and $\overline{\triangle} := \overline{\triangle}(\overline{x},\overline{y},\overline{z})$, and for any two points $p,q \in \triangle$ and these comparison points $\overline{p}, \overline{q} \in \overline{\triangle}$, an inequality $d(p,q) \leq d'(\overline{p}, \overline{q})$ always holds. That inequality $d(p,q) \leq d'(\overline{p}, \overline{q})$ is often called a CAT($\kappa$) inequality.

A CAT($\kappa$) space $X$ is said to be *admissible* if $d(x,y) < D_\kappa/2$ holds for every $x, y \in X$. Since $D_\kappa = \infty$ if $\kappa \leq 0$, CAT($\kappa$) spaces are clearly admissible if $\kappa \leq 0$. Note that every CAT($\kappa$) space is also a CAT($\kappa'$) space if $\kappa < \kappa'$ [2].

A Hilbert space is one of the complete CAT(0) space, and therefore is a complete CAT($\kappa$) space for any $\kappa \geq 0$. It means that the class of the complete CAT(0) spaces includes Hilbert spaces, but not Banach spaces in general. Moreover, we easily obtain that a model space $M_\kappa$ is a complete CAT($\kappa$) space.

Let $X$ be a CAT($\kappa$) space and $C$ a nonempty subset of $X$. Then a convex hull of $C$, which is written as $\operatorname{co} C$, is defined by $\bigcup_{n=1}^{\infty} C_n$, where $C_1 = C$ and $C_{n+1} = \{[x,y] \mid x, y \in C_n\}$ for every $n \in \mathbb{N}$. Moreover, $\operatorname{cl} C$ denotes a closure of $C$. A complete CAT($\kappa$) space $X$ is said to have the *convex hull finite property (CHFP)* if for any nonempty finite subset $E$ of $X$ and every continuous mapping $T$ from $\operatorname{cl}\operatorname{co} E$ into itself, $T$ has a fixed point.

Let $X$ be a CAT($\kappa$) space and let $t \in [0,1]$. Then, the following inequalities hold for any $x, y, z \in X$ with $d(x,y) + d(y,z) + d(z,x) < 2D_\kappa$:

- If $\kappa > 0$,

$$\begin{aligned}&\cos(\sqrt{\kappa}\, d(tx \oplus (1-t)y, z)) \sin(\sqrt{\kappa}\, D)\\ &\quad \geq \cos(\sqrt{\kappa}\, d(x,z)) \sin(t\sqrt{\kappa}\, D) + \cos(\sqrt{\kappa}\, d(y,z)) \sin((1-t)\sqrt{\kappa}\, D);\end{aligned}$$

- if $\kappa = 0$,

$$d(tx \oplus (1-t)y, z)^2 \leq td(x,z)^2 + (1-t)d(y,z)^2 - t(1-t)d(x,y)^2;$$

- if $\kappa < 0$,

$$\begin{aligned}&\cosh(\sqrt{\kappa}\, d(tx \oplus (1-t)y, z)) \sinh(\sqrt{\kappa}\, D)\\ &\quad \leq \cosh(\sqrt{\kappa}\, d(x,z)) \sinh(t\sqrt{\kappa}\, D) + \cosh(\sqrt{\kappa}\, d(y,z)) \sinh((1-t)\sqrt{\kappa}\, D),\end{aligned}$$

where $D = d(x,y)$. These inequalities are called the *parallelogram law* on CAT($\kappa$) spaces. These hold as an equation if CAT($\kappa$) space $X$ is just a model space $M_\kappa$, see [2].

The following are easily obtained by a parallelogram law on a CAT($\kappa$) space $X$:

- If $\kappa > 0$,

$$\cos(\sqrt{\kappa}\, d(tx \oplus (1-t)y, z)) \geq t\cos(\sqrt{\kappa}\, d(x,z)) + (1-t)\cos(\sqrt{\kappa}\, d(y,z));$$

- if $\kappa = 0$,

$$d(tx \oplus (1-t)y, z)^2 \leq td(x,z)^2 + (1-t)d(y,z)^2;$$

- if $\kappa < 0$,

$$\cosh(\sqrt{\kappa}\, d(tx \oplus (1-t)y, z)) \leq t\cosh(\sqrt{\kappa}\, d(x,z)) + (1-t)\cosh(\sqrt{\kappa}\, d(y,z))$$

for any $x, y, z \in X$ with $d(x,z) < D_\kappa/2$, $d(y,z) < D_\kappa/2$, $d(x,y) < D_\kappa$, and $t \in [0,1]$. In this paper, we call these inequalities the *corollaries of the parallelogram law* on CAT($\kappa$) spaces.

Let $X$ be a CAT($\kappa$) space and $f$ a real function on $X$. Then $f$ is said to be *convex* if for any $x, y \in X$ and $t \in\ ]0,1[$, $f(tx \oplus (1-t)y) \leq tf(x) + (1-t)f(y)$ holds, and is said to be *upper hemicontinuous* if $\limsup_{t \downarrow 0} f(tx \oplus (1-t)y) \leq f(y)$ for any $x, y \in X$.

### *2.2. A function $c_\kappa$*

For $\kappa \in \mathbb{R}$, define a function $c_\kappa \colon ]-\infty, \infty] \to [0, \infty]$ by

$$c_\kappa(d) = \sum_{n=1}^{\infty} \frac{\kappa^{n-1}(-1)^{n-1}d^{2n}}{(2n)!} = \begin{cases} \dfrac{1}{-\kappa}(\cosh(\sqrt{-\kappa}\, d) - 1) & (\text{if } \kappa < 0); \\ \dfrac{1}{2}d^2 & (\text{if } \kappa = 0); \\ \dfrac{1}{\kappa}(1 - \cos(\sqrt{\kappa}\, d)) & (\text{if } \kappa > 0) \end{cases}$$

for $d \in \mathbb{R}$ and $c_\kappa(\infty) = \infty$. Then the function $c_\kappa$ is infinitely differentiable on $\mathbb{R}$ and we get

$$c'_\kappa(d) = \sum_{n=1}^{\infty} \frac{\kappa^{n-1}(-1)^{n-1}d^{2n-1}}{(2n-1)!} = \begin{cases} \dfrac{1}{\sqrt{-\kappa}} \sinh(\sqrt{-\kappa}\, d) & (\text{if } \kappa < 0); \\ d & (\text{if } \kappa = 0); \\ \dfrac{1}{\sqrt{\kappa}} \sin(\sqrt{\kappa}\, d) & (\text{if } \kappa > 0) \end{cases}$$

and

$$c''_\kappa(d) = \sum_{n=0}^{\infty} \frac{\kappa^{n}(-1)^{n}d^{2n}}{(2n)!} = 1 - \kappa c_\kappa(d) = \begin{cases} \cosh(\sqrt{-\kappa}\, d) & (\text{if } \kappa < 0); \\ 1 & (\text{if } \kappa = 0); \\ \cos(\sqrt{\kappa}\, d) & (\text{if } \kappa > 0) \end{cases}$$

for $d \in \mathbb{R}$. Then for any $\kappa \in \mathbb{R}$, we get the following facts:

- $c_\kappa(0) = 0$, and $c_\kappa(d) > 0$ for any $d \in ]0, D_\kappa/2[$,
- $c'_\kappa(0) = 0$, and $c'_\kappa(d) > 0$ for any $d \in ]0, D_\kappa[$,
- $c''_\kappa(0) = 1$, and $c''_\kappa(d) > 0$ for any $d \in ]0, D_\kappa/2[$,
- $c_\kappa$ is an odd function, and $c''_\kappa$ is an even function,
- $c_\kappa$ is convex on $[0, D_\kappa/2]$,
- $c''_\kappa(d)^2 + \kappa c'_\kappa(d)^2 = 1$ for any $d \in \mathbb{R}$.

For each $t \in [0, 1]$, $d \in [0, D_\kappa[$ and $\kappa \in \mathbb{R}$, put

$$(t)^\kappa_d = \begin{cases} \dfrac{c'_\kappa(td)}{c'_\kappa(d)} & (\text{if } d \neq 0); \\ t & (\text{if } d = 0). \end{cases}$$

Then we have

$$(t)^\kappa_d = \begin{cases} \sinh(t\sqrt{-\kappa}\, d)/\sinh(\sqrt{-\kappa}\, d) & (\text{if } \kappa < 0); \\ t & (\text{if } \kappa = 0); \\ \sin(t\sqrt{\kappa}\, d)/\sin(\sqrt{\kappa}\, d) & (\text{if } \kappa > 0) \end{cases}$$

for any $d \in ]0, D_\kappa[$. Kimura and Sudo [10] discovered that we can write all parallelogram laws on CAT($\kappa$) spaces in the same formula as follows:

$$c_\kappa(d(tx \oplus (1-t)y, z)) \leq (t)^\kappa_D(c_\kappa(d(x, z)) - c_\kappa((1-t)D)) + (1-t)^\kappa_D(c_\kappa(d(y, z)) - c_\kappa(tD))$$

for any $x, y, z \in X$ with $d(x, y) + d(y, z) + d(z, x) < 2D_\kappa$ and $t \in [0, 1]$, where $D = d(x, y)$. Similarly, the corollaries of the parallelogram law on a CAT($\kappa$) space $X$ can be written by

$$c_\kappa(d(tx \oplus (1-t)y, z)) \leq tc_\kappa(d(x, z)) + (1-t)c_\kappa(d(y, z))$$

for any $x, y, z \in X$ with $d(x, z) < D_\kappa/2$, $d(y, z) < D_\kappa/2$, $d(x, y) < D_\kappa$, and $t \in [0, 1]$.

### *2.3. Vicinal mappings*

The notion of vicinal mappings is first proposed by Kohsaka [12]. Motivated by this paper, Kajimura and Kimura [5] proposed the notion of vicinal mapping with $\psi$ in 2019. Let $X$ be an admissible CAT($\kappa$) space and take $\psi\colon [0, D_\kappa/2[ \to ]0,\infty[$ such that $\psi$ is right continuous at 0. A mapping $T\colon X \to X$ is said to be *vicinal with* $\psi$ if

$$\big(\psi(d(x,Tx)) + \psi(d(y,Ty))\big)c_\kappa(d(Tx,Ty)) \leq \psi(d(x,Tx))c_\kappa(d(x,Ty)) + \psi(d(y,Ty))c_\kappa(d(y,Tx))$$

holds for any $x, y \in X$. A mapping $T\colon X \to X$ is said to be *firmly vicinal with* $\psi$ if

$$\begin{aligned}\big(\psi(d(x,Tx))c_\kappa(d(x,Tx)) + \psi(d(y,Ty))c_\kappa(d(y,Ty))\big)c''_\kappa(d(Tx,Ty))&\\ + \big(\psi(d(x,Tx)) + \psi(d(y,Ty))\big)c_\kappa(d(Tx,Ty))&\\ \leq \psi(d(x,Tx))c_\kappa(d(x,Ty)) + \psi(d(y,Ty))c_\kappa(d(y,Tx))&\end{aligned}$$

holds for any $x, y \in X$. It can be easily obtained that every firmly vicinal mapping with $\psi$ is vicinal with $\psi$.

**Lemma 2.1** (Kajimura and Kimura [4]). *Let $X$ be a complete* CAT(0) *space and $f$ a proper lower semicontinuous convex coercive function from $X$ into $]-\infty,\infty]$. Suppose that*

$$f\left(\frac{1}{2}y_1 \oplus \frac{1}{2}y_2\right) < \frac{1}{2}f(y_1) + f(y_2)$$

*for any $y_1, y_2 \in X$ with $y_1 \neq y_2$. Then $f$ has the unique minimizer on $X$.*

**Lemma 2.2** (Kajimura and Kimura [5]). *Let $X$ be an admissible* CAT($\kappa$) *space. Suppose that $T\colon X \to X$ is vicinal with some $\psi$. Then $T$ is $\Delta$-demiclosed. Moreover, if $F(T)$ is nonempty, then $T$ is quasinonexpansive.*

**Lemma 2.3** (Kajimura and Kimura [5]). *Let $X$ be an admissible* CAT($\kappa$) *space. Suppose that $T\colon X \to X$ is firmly vicinal with some $\psi$. If $F(T)$ is nonempty, then $T$ is asymptotically regular.*

**Lemma 2.4** (Kimura and Sasaki [9]). *Let $X$ be an admissible* CAT($\kappa$) *space and let $\psi\colon [0,\infty[ \to ]0,\infty[$ such that $\psi$ is right continuous at* 0*. Then for a mapping $T\colon X \to X$, the following are equivalent:*

(i) *$T$ is firmly vicinal with $\psi$,*

(ii) *for any $x, y \in X$,*

$$\begin{aligned}&\big(\psi(d(x,Tx))c''_\kappa(d(x,Tx)) + \psi(d(y,Ty))c''_\kappa(d(y,Ty))\big)\, c_\kappa(d(Tx,Ty))\\ &\quad\leq \psi(d(x,Tx))\big(c_\kappa(d(x,Ty)) - c_\kappa(d(x,Tx))\big) + \psi(d(y,Ty))\big(c_\kappa(d(y,Tx)) - c_\kappa(d(y,Ty))\big),\end{aligned}$$

(iii) *for any $x, y \in X$,*

$$\begin{aligned}&\frac{1}{\kappa}\big(\psi(d(x,Tx))c''_\kappa(d(x,Tx)) + \psi(d(y,Ty))c''_\kappa(d(y,Ty))\big)c''_\kappa(d(Tx,Ty))\\ &\qquad\geq \frac{1}{\kappa}\big(\psi(d(x,Tx))c''_\kappa(d(x,Ty)) + \psi(d(y,Ty))c''_\kappa(d(y,Tx))\big),\end{aligned}$$

*where* (iii) *is considered only when $\kappa \neq 0$.*

### *2.4. Equilibrium problems*

Let $X$ be a uniquely $D$-geodesic space and $K$ a nonempty closed convex subset of $X$. The equilibrium problem for $f\colon K^2 \to \mathbb{R}$ is a problem to find a point $z \in K$ satisfying $\inf_{y\in K} f(z,y) \geq 0$. For a bifunction $f$, let us denote the set of all solutions to the equilibrium problem by $\operatorname{Equil} f$, that is, $\operatorname{Equil} f = \{z \in K \mid \inf_{y\in K} f(z,y) \geq 0\}$. In this paper, we always assume that $f$ satisfies the following conditions (E1)–(E4):

(E1) $f(z,z) = 0$ for all $z \in K$;

(E2) $f(z,y) + f(y,z) \leq 0$ for all $z, y \in K$;

(E3) $f(z,\cdot)\colon K \to \mathbb{R}$ is lower semicontinuous and convex for all $z \in K$;

(E4) $f(\cdot,y)\colon K \to \mathbb{R}$ is upper hemicontinuous for all $y \in K$.

## 3. Main results

To show the main result, we use the following lemma.

**Lemma 3.1** (Kimura and Sasaki [9]). *Let $X$ be an admissible complete* CAT($\kappa$) *space and suppose that $X$ has the convex hull finite property. Let $K$ be a nonempty closed convex subset of $X$ and $f$ a real function on $K^2$ with conditions* (E1)–(E4). *Let $\varphi\colon [0, c_\kappa(D_\kappa/2)[ \to [0,\infty[$ be a strictly increasing and differentiable function such that $\varphi'$ is continuous. Define $\Phi\colon [0, D_\kappa/2[ \to [0,\infty[$ by $\Phi d = \varphi(c_\kappa(d))$ for $d \in [0, D_\kappa/2[$. Suppose that $\Phi$ is convex. In addition, assume the following:*

- *If $\kappa \le 0$ and $K$ is unbounded, then assume*

$$\liminf_{\substack{d(v,z)\to\infty \\ z\in K}} \frac{f(v,z)}{d(v,z)} + \liminf_{d\to\infty} \frac{\Phi d}{d} > 0$$

  *for all $v \in K$;*
- *if $\kappa > 0$, then assume $\lim_{d\to D_\kappa/2} \Phi d = \infty$;*
- *otherwise, give no additional assumptions.*

*Define a subset $R_f x$ of $K$ by*

$$R_f x = \left\{ z \in K \;\middle|\; \inf_{y\in K} (f(z,y) + \Phi d(x,y) - \Phi d(x,z)) \ge 0 \right\}$$

*for each $x \in X$. Then $R_f x$ is nonempty for any $x \in X$. Moreover,*

$$0 \le f(z,w) + \varphi'(c_\kappa(d(x,z))) \cdot \frac{D}{c'_\kappa(D)} \big( c_\kappa(d(x,w)) - c_\kappa(D) - c''_\kappa(D) c_\kappa(d(x,z)) \big)$$

*holds for any $z \in R_f x$ and $w \in K \setminus \{z\}$, where $D = d(z,w)$.*

Now we show the main result.

**Theorem 3.2.** *For $\kappa \le 0$, let $X$ be an admissible complete* CAT($\kappa$) *space and suppose that $X$ has the convex hull finite property. Let $K$ be a nonempty closed convex subset of $X$ and $f$ a real function on $K^2$ with conditions* (E1)–(E4). *Let $\varphi\colon [0,\infty[ \to [0,\infty[$ be a strictly increasing and differentiable function such that $\varphi'$ is continuous. Define $\Phi\colon [0,\infty[ \to [0,\infty[$ by $\Phi d = \varphi(c_\kappa(d))$ for $d \in [0,\infty[$. If $K$ is unbounded, then assume*

$$\liminf_{\substack{d(v,z)\to\infty \\ z\in K}} \frac{f(v,z)}{d(v,z)} + \liminf_{d\to\infty} \frac{\Phi d}{d} > 0$$

*for any $v \in K$. Furthermore, suppose the following two conditions:*

(a) *$\Phi = \varphi \circ c_\kappa$ is convex on $[0,\infty[$;*

(b) *$\Phi d(x,\cdot)$ is strictly midpoint convex on $K$ for any $x \in X$, namely, an inequality*

$$\Phi d\left(x, \frac{1}{2}y_1 \oplus \frac{1}{2}y_2\right) < \frac{1}{2}\Phi d(x,y_1) + \frac{1}{2}\Phi d(x,y_2)$$

*holds for any $x \in X$ and $y_1, y_2 \in K$ with $y_1 \ne y_2$.*

*Define a set-valued mapping $R_f\colon X \to 2^K$ by*

$$R_f x = \left\{ z \in K \;\middle|\; \inf_{y\in K} (f(z,y) + \Phi d(x,y) - \Phi d(x,z)) \ge 0 \right\}$$

*for each $x \in X$. Then the following hold:*

(i) *$R_f x$ consists of one point for every $x \in X$, and thus $R_f\colon X \to K$ is defined as a single-valued mapping;*

(ii) *$R_f$ satisfies the following inequality for any $x_1, x_2 \in X$:*

$$\begin{aligned}&\left(\psi(D_1)c_\kappa''(D_1) + \psi(D_2)c_\kappa''(D_2)\right) c_\kappa(d(R_f x_1, R_f x_2))\\ &\qquad\le \psi(D_1)\left(c_\kappa(d(x_1, R_f x_2)) - c_\kappa(D_1)\right) + \psi(D_2)\left(c_\kappa(d(x_2, R_f x_1)) - c_\kappa(D_2)\right),\end{aligned}$$

*where $\psi := \varphi' \circ c_\kappa$, $D_1 := d(x_1, R_f x_1)$, and $D_2 := d(x_2, R_f x_2)$;*

(iii) *$R_f$ is firmly vicinal with $\psi = \varphi' \circ c_\kappa$;*

(iv) *$R_f$ is $\Delta$-demiclosed; if $F(R_f) \neq \emptyset$, then $R_f$ is quasinonexpansive and asymptotically regular;*

(v) *$F(R_f) = \operatorname{Equil} f$ holds, and thus $\operatorname{Equil} f$ is closed and convex.*

*Proof.* By Lemma 3.1, $R_f x$ is nonempty for any $x \in X$.

(i) Take $x \in X$ and $z \in K$ arbitrarily, and put $g_z(\cdot) = f(z, \cdot) + \Phi d(x, \cdot)$. Then $g_z\colon K \to \mathbb{R}$ is lower semicontinuous and convex. By the assumptions for $f$ and $\Phi$, we get

$$\begin{aligned} g_z\left(\frac{1}{2}y_1 \oplus \frac{1}{2}y_2\right) &= f\left(z, \frac{1}{2}y_1 \oplus \frac{1}{2}y_2\right) + \Phi d\left(x, \frac{1}{2}y_1 \oplus \frac{1}{2}y_2\right)\\ &< \frac{1}{2}g_z(y_1) + \frac{1}{2}g_z(y_2)\end{aligned}$$

for any $y_1, y_2 \in K$ with $y_1 \neq y_2$. Moreover, we obtain

$$\begin{aligned}\liminf_{\substack{d(z,y)\to\infty\\ y\in K}} \frac{g_z(y)}{d(z,y)} &\ge \liminf_{\substack{d(z,y)\to\infty\\ y\in K}} \frac{f(z,y)}{d(z,y)} + \liminf_{\substack{d(z,y)\to\infty\\ y\in K}} \frac{\Phi d(x,y)}{d(z,y)}\\ &= \liminf_{\substack{d(z,y)\to\infty\\ y\in K}} \frac{f(z,y)}{d(z,y)} + \liminf_{d\to\infty} \frac{\Phi d}{d} > 0\end{aligned}$$

and hence $g_z(y) \to \infty$ if $d(z, y) \to \infty$. It concludes that $g_z$ has the unique minimizer by Lemma 2.1.

Put $y_z = \operatorname{argmin}_{y\in K} g_z(y)$ for each $z \in K$. Let $z_1, z_2 \in R_f x$. Then we have $f(z_1, y_{z_1}) + \Phi d(x, y_{z_1}) - \Phi d(x, z_1) \ge 0$ and $f(z_2, y_{z_2}) + \Phi d(x, y_{z_2}) - \Phi d(x, z_2) \ge 0$. Thus

$$f(z_1, y_{z_1}) + f(z_2, y_{z_2}) \ge \Phi d(x, z_1) + \Phi d(x, z_2) - \Phi d(x, y_{z_1}) - \Phi d(x, y_{z_2})$$

holds. Assume that $z_1 \neq z_2$. Then we obtain

$$\begin{aligned} f(z_1, y_{z_1}) + \Phi d(x, y_{z_1}) &\le f\left(z_1, \frac{1}{2}z_1 \oplus \frac{1}{2}z_2\right) + \Phi d\left(x, \frac{1}{2}z_1 \oplus \frac{1}{2}z_2\right)\\ &< \frac{1}{2}f(z_1, z_2) + \frac{1}{2}\Phi d(x, z_1) + \frac{1}{2}\Phi d(x, z_2)\end{aligned}$$

and similarly we get

$$f(z_2, y_{z_2}) + \Phi d(x, y_{z_2}) < \frac{1}{2}f(z_2, z_1) + \frac{1}{2}\Phi d(x, z_2) + \frac{1}{2}\Phi d(x, z_1).$$

Summing up these inequalities, we obtain

$$f(z_1, y_{z_1}) + f(z_2, y_{z_2}) < \Phi d(x, z_1) + \Phi d(x, z_2) - \Phi d(x, y_{z_1}) - \Phi d(x, y_{z_2}),$$

which is a contradiction. Thus $R_f x$ is a singleton for every $x \in X$.

(ii) It is obtained by the same calculation as in [9].

(iii) The inequality (ii) is exactly equivalent to the firm vicinality with $\varphi' \circ c_\kappa$ of $R_f$, from Lemma 2.4.

(iv) By Lemma 2.2 and Lemma 2.3, we get (iv).

(v) Let $z \in F(R_f)$, $y \in K$, and put $D = d(z, y)$. Then we get

$$\begin{aligned} 0 &\le f(z,y) + \varphi'(c_\kappa(d(x,z))) \cdot \frac{D}{c'_\kappa(D)}\big(c_\kappa(d(R_f z, y)) - c_\kappa(D) - c''_\kappa(D)c_\kappa(d(R_f z, z))\big) \\ &= f(z,y). \end{aligned}$$

Thus $f(z, y) \ge 0$ if $z \ne y$, and it holds even if $z = y$. It implies that $\inf_{y\in K} f(z, y) \ge 0$ and therefore $F(R_f) \subset \operatorname{Equil} f$ holds.

Next, suppose that $z \in \operatorname{Equil} f$. Then we have

$$\inf_{y\in K}(f(z,y) + \Phi d(z,y) - \Phi d(z,z)) \ge \inf_{y\in K} f(z,y) \ge 0$$

and thus $z = R_f z$, which implies $F(R_f) = \operatorname{Equil} f$.

Finally, suppose that $F(R_f) \ne \emptyset$. Then $R_f$ is quasinonexpansive by (iv), and it follows that $F(R_f)$ is closed and convex. Since $F(R_f) = \operatorname{Equil} f$, we get the conclusion. ☐

This result means that we can define the resolvent operator $R_f$ by using the perturbation function $\Phi$ with the convexity of $\Phi$ and the strict midpoint convexity of $\Phi d(x, \cdot)$ on $K$ instead of the nondecreasingness of $\varphi'$.

From this result, we can show that the resolvent for the equilibrium problem defined by using the perturbation $\Phi\colon d \mapsto \tanh d \sinh d$ is well-defined on complete CAT$(-1)$ spaces. Now we prepare a lemma.

**Lemma 3.3** (Kajimura and Kimura [4]). *Let $f\colon X \to\, ]-\infty, \infty]$ be a proper lower semicontinuous convex function. Take $p \in X$ and define a function $g\colon X \to\, ]-\infty, \infty]$ by $g(\cdot) = f(\cdot) + \tanh d(\cdot, p) \sinh d(\cdot, p)$. Then $g$ is strictly midpoint convex, that is,*

$$g\left(\frac{1}{2}x \oplus \frac{1}{2}y\right) < \frac{1}{2}g(x) + \frac{1}{2}g(y)$$

*holds for any $x, y \in X$ with $x \ne y$.*

**Corollary 3.4.** *A function $g\colon X \to\, ]-\infty, \infty]$ defined by*

$$g(\cdot) = \tanh d(\cdot, p) \sinh d(\cdot, p)$$

*is strictly midpoint convex for any $p \in X$.*

From this lemma and Theorem 3.2, we get the desired result as follows.

**Corollary 3.5.** *Let $X$ be a complete* CAT$(-1)$ *space and suppose that $X$ has the convex hull finite property. Let $K$ be a nonempty closed convex subset of $X$ and $f$ a real function on $K^2$ with conditions* (E1)–(E4). *Then a set-valued operator $S_f\colon X \to 2^K$ defined by*

$$S_f x = \left\{ z \in K \;\middle|\; \inf_{y\in K}\big(f(z,y) + \tanh d(x,y)\sinh d(x,y) - \tanh d(x,z)\sinh d(x,z)\big) \ge 0 \right\}$$

*for every $x \in X$ is well-defined as a single-valued mapping from $X$ into $K$. Furthermore, the following hold.*

(i) *$R_f$ is firmly vicinal with $\psi\colon d \mapsto 1 + 1/\cosh^2 d$. Namely, $R_f$ satisfies*

$$\cosh d(R_f x_1, R_f x_2) \le \frac{\left(\frac{1}{C_1^2}+1\right)\cosh d(x_1, R_f x_2) + \left(\frac{1}{C_2^2}+1\right)\cosh d(x_2, R_f x_1)}{\left(\frac{1}{C_1^2}+1\right)\cosh d(x_1, R_f x_1) + \left(\frac{1}{C_2^2}+1\right)\cosh d(x_2, R_f x_2)}$$

*for any $x_1, x_2 \in X$, where $C_1 = \cosh d(x_1, R_f x_1)$ and $C_2 = \cosh d(x_2, R_f x_2)$.*

(ii) $R_f$ *is hyperbolically nonspreading.*

*Proof.* (i) Let $\kappa = -1$ and $\varphi(t) = t + 1 - 1/(t+1)$ for $t \in [0, \infty[$. Then, since $c_\kappa(d) = c_{-1}(d) = \cosh d - 1$ for $d \in [0, \infty[$, we have $(\varphi \circ c_\kappa)(d) = \tanh d \sinh d$ for any $d \in [0, \infty[$. Therefore, a function $\Phi = \varphi \circ c_\kappa$ is convex on $[0, \infty[$. Moreover, we obtain that $\Phi d(x, \cdot)$ is strictly midpoint convex on $K$ for any $x \in X$ from Corollary 3.4. It deduces that $S_f x$ consists exactly one point for any $x \in X$ from Theorem 3.2. Since $\varphi'(t) = 1 + 1/(t+1)^2$ for any $t \in [0, \infty[$, we get $S_f$ is firmly vicinal with $\psi \colon d \mapsto (\varphi' \circ c_{-1})(d) = 1 + 1/\cosh^2 d$.

(ii) Put $C_1 = \cosh d(x_1, R_f x_1)$ and $C_2 = \cosh d(x_2, R_f x_2)$. Then we have

$$\begin{aligned}
\cosh d(R_f x_1, R_f x_2) &\le \frac{\left(\frac{1}{C_1^2} + 1\right) \cosh d(x_1, R_f x_2) + \left(\frac{1}{C_2^2} + 1\right) \cosh d(x_2, R_f x_1)}{\left(\frac{1}{C_1^2} + 1\right) C_1 + \left(\frac{1}{C_2^2} + 1\right) C_2^2} \\
&= \frac{\left(\frac{1}{C_1^2} + 1\right) \cosh d(x_1, R_f x_2) + \left(\frac{1}{C_2^2} + 1\right) \cosh d(x_2, R_f x_1)}{(C_1 + C_2)\left(1 + \frac{1}{C_1 C_2}\right)} \\
&\le \frac{2 \cosh d(x_1, R_f x_2) + 2 \cosh d(x_2, R_f x_1)}{2\sqrt{C_1 C_2} \cdot 2\sqrt{1 \cdot \frac{1}{C_1 C_2}}} \\
&= \frac{\cosh d(x_1, R_f x_2) + \cosh d(x_2, R_f x_1)}{2},
\end{aligned}$$

which means that $R_f$ is hyperbolically nonspreading. □

## 4. Applications

In the previous section, we prove the well-definedness of a resolvent $S_f$ of the equilibrium problem which is defined by using a perturbation function $\Phi \colon d \mapsto \tanh d \sinh d$ on complete CAT$(-1)$ spaces. We also show that $S_f$ has the hyperbolical nonspreadingness.

In 2019, Kajimura and Kimura [4] showed the following.

**Lemma 4.1** (Kajimura and Kimura [4])**.** *Let $X$ be a complete* CAT$(-1)$ *space and $T$ a hyperbolically nonspreading mapping from $X$ into itself. Suppose that a sequence $\{T^n y\}$ is bounded for some $y \in X$. Then, $T$ has a fixed point. Furthermore, for any initial point $x \in X$, a sequence $\{T^n x\}$ $\Delta$-converges to a fixed point of $T$.*

The concept of $\Delta$-convergence, which corresponds to the weak convergence on Hilbert spaces, is defined as follows. Let $X$ be a complete CAT$(-1)$ space. A point $z \in X$ is called an *asymptotic center* of a bounded sequence $\{x_n\}$ if $z$ is a minimizer of a function $\limsup_{n\to\infty} d(\cdot, x_n)$ on $X$. A bounded sequence $\{x_n\}$ is said to $\Delta$-converge to a point $z \in X$ if $z$ is the unique asymptotic center of any subsequence of $\{x_n\}$. Note that the $\Delta$-limit of any $\Delta$-convergent sequence $\{x_n\}$ is always unique, and write it as $\Delta\text{-}\lim_{n\to\infty} x_n$. In addition, if $\{x_n\}$ is convergent, then we get $\lim_{n\to\infty} x_n = \Delta\text{-}\lim_{n\to\infty} x_n$.

Using this lemma and Corollary 3.5, we get the following result.

**Theorem 4.2.** *Let $X$ be a complete* CAT$(-1)$ *space and suppose that $X$ has the convex hull finite property. Let $K$ be a nonempty closed convex subset of $X$ and $f$ a real function on $K^2$ with conditions* (E1)–(E4)*. Define a resolvent $S_f \colon X \to K$ by*

$$S_f x = \left\{ z \in K \;\middle|\; \inf_{y \in K} \big( f(z, y) + \tanh d(x, y) \sinh d(x, y) - \tanh d(x, z) \sinh d(x, z) \big) \ge 0 \right\}$$

*for every $x \in X$. Suppose that a sequence $\{S_f^n y\}$ is bounded for some $y \in X$. Then, $S_f$ has a fixed point, therefore $f$ has a solution to its equilibrium problem. Furthermore, a sequence $\{S_f^n x\}$ $\Delta$-converges to an element of* Equil $f$ *for every $x \in X$.*

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## References

[1] E. Blum and W. Oettli, *From optimization and variational inequalities to equilibrium problems*, Math. Stud. **63** (1994), 123–145.

[2] M. R. Bridson and A. Haefliger, *Metric Spaces of Non–Positive Curvature*, vol. 319 of *Grundlehren der. Mathematischen Wissenschaften*, Springer, Berlin, 1999.

[3] P. L. Combettes and S. A. Hirstoaga, *Equilibrium programming in Hilbert spaces*, J. Nonlinear Convex Anal. **6** (2005), 117–136.

[4] T. Kajimura and Y. Kimura, *Resolvents of convex functions in complete geodesic metric spaces with negative curvature*, J. Fixed Point Theory Appl. **21**, no. 32 (2019), 15 pages.

[5] T. Kajimura and Y. Kimura, *A vicinal mapping on geodesic spaces*, Proceedings of the International Conference on Nonlinear Analysis and Convex Analysis & International Conference on Optimization: Techniques and Applications -I- (Hakodate, Japan, 2019), 183–195.

[6] Y. Kimura, *Resolvents of equilibrium problems on a complete geodesic space with curvature bounded above*, Carpathian J. Math. **37** (2021), 463–476.

[7] Y. Kimura and Y. Kishi, *Equilibrium problems and their resolvents in Hadamard spaces*, J. Nonlinear Convex Anal. **19** (2018), 1503–1513.

[8] Y. Kimura and F. Kohsaka, *Spherical nonspreadingness of resolvents of convex functions in geodesic spaces*, J. Fixed Point Theory Appl. **18** (2016), 93–115.

[9] Y. Kimura and K. Sasaki, *Sufficient conditions for perturbations to define the resolvent of the equilibrium problem on complete geodesic spaces*, J. Fixed Point Theory Appl, **25**, no. 76 (2023), 24 pages.

[10] Y. Kimura and S. Sudo, *New type parallelogram laws in Banach spaces and geodesic spaces with curvature bounded above*, Arab. J. Math. (2022).

[11] Y. Kimura and T. Ogihara, *Resolvents of equilibrium problems in a complete geodesic space with negative curvature*, arXiv:2207.10903v1 [math.FA] (2022).

[12] F. Kohsaka, *Existence and approximation of fixed points of vicinal mappings in geodesic spaces*, Pure Appl. Funct. Anal. **3** (2018), 91–106.

[13] U. F. Mayer, *Gradient flows on nonpositively curved metric spaces a d harmonic maps*, Commun. Anal. Geom. **6** (1998), 199–206.