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A New Generalized Family of Odd Lomax-G Distributions Properties and Applications

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Abstract

The motivation for this study is to develop the new family of continuous distributions called Odd Lomax-G (OLG). Also present a new flexible three-parameter distribution according to the developed family called the Odd Lomax-G Exponential (OLE) distribution. Using binomial series expansion, logarithmic, and exponential expansions, the new OLG family and OLE distribution are expanded. We find the derivative of the moments, the mgf, quantity function, ordered statistics and Rényi entropy. Then use the MLE method to estimate the OLE model parameters. Finally, an importance of the new family is made clear experimentally through two real data applications. Then explore the performance of OLE distribution inferred from family OLG based on certain goodness of fit criteria.

Keywords: OLG family; OLE distribution; Quantity function; Ordered statistics; Rényi entropy MLE_s method.
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1. Introduction

Statistical distributions are often used to describe and forecast natural phenomena. Many statistical distributions exist. However, there was a need to create flexible distributions. These would accurately describe data, natural phenomena, or limited real-world scenarios. Many researchers in statistics introduced new distributions like the Go-G family [1], MOG family [2], and MOTL-G family [3]. There were also the APRAY-G family [4], MOW-G family [5], TIIEHL-Gom-TL-G family [6], NETLE family [7], Muth-G family [8], and TNH-G family [9].

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The steps to derive our family of distributions, which called A New Generalized Family of Odd Lomax-G Distributions in short (OLG), include: Let

$$W(G(x,\xi)) = -G(x,\xi).\log(1-G(x,\xi))$$

Where $W\left(G\left(x,\xi\right)\right)$ is CDF satisfying the conditions?

1. $W\left(G\left(x,\xi\right)\right) \in [a,b], -\infty < a < b < \infty$
2. $W\left(G\left(x,\xi\right)\right)$ is differentiable and monotonically non-decreasing.
3. $W \to b$, as $x \to \left(G\left(x,\xi\right)\right) \to a,\ as\ x \to -\infty$, and

$$W\left(G\left(x,\xi\right)\right)\infty \tag{1}$$

i.e. $W\left(G\left(x,\xi\right)\right) \to 0,\ as\ x \to 0$, and $W\left(G\left(x,\xi\right)\right) \to 1$, as $x \to \infty$

Alzaatreh in 2013 proposed a new method to generate new CDF of $T-X$ family distributions by [10]:

$$F\left(x,\xi\right) = \int_0^{W(G(x,\xi))} r\left(u\right)du \tag{2}$$

Where $F\left(x,\xi\right)$ CDF of A new family is, $r\left(u\right)$ is pdf of random variableu $\in [a,b], W\left(G\left(x,\xi\right)\right)$ satisfy Eq.(1), take $r\left(x\right)$, $R\left(x\right)$ are PDF and CDF of Lomax distribution respectively given by form:

$$r\left(x\right) = \frac{\alpha}{\gamma}\left(1+\frac{x}{\gamma}\right)^{-(\alpha+1)}, x > 0, \alpha, \gamma > 0 \tag{3}$$

$$R\left(x\right) = 1-\left(1+\frac{x}{\gamma}\right)^{-\alpha} x > 0,\ \alpha, \gamma > 0 \tag{4}$$

Then the CDF of OLG family is given by:

$$F_{OL}\left(x,\alpha,\gamma,\xi\right) = \int_0^{-G(x,\xi).log(1-G(x,\xi))} \frac{\alpha}{\gamma}\left(1+\frac{u}{\gamma}\right)^{-(\alpha+1)} du$$

$$F_{OLoG}\left(x,\alpha,\gamma,\xi\right) = 1-\left(1-\frac{G\left(x,\xi\right).log(1-G\left(x,\xi\right))}{\gamma}\right)^{-\alpha} \tag{5}$$

While pdf of OLG family from Eq.(5) is:

$$\begin{aligned} f_{OLoG}\left(x,\alpha,\gamma,\xi\right) = &\frac{\alpha}{\gamma}g\left(x,\xi\right)\left(1-\frac{G\left(x,\xi\right).\log\left(1-G\left(x,\xi\right)\right)}{\gamma}\right)^{-(\alpha+1)} \\ &\left[\frac{G\left(x,\xi\right)}{1-G\left(x,\xi\right)} - \log\left(1-G\left(x,\xi\right)\right)\right], x, \alpha, \gamma, \xi > 0 \end{aligned} \tag{6}$$

The study aims to introduce a new distribution family and describe some of its features. This family's structure combines the individual Lomax distribution with $T-X$. The proposed OLG family is a better fit than some known distributions. This distribution family is important because it describes how data sets decrease and increase. The second goal is to present the exponential distribution baseline. It also includes statistical characteristics of the distribution based on the proposed family. Additionally, it involves using MLEs to estimate the model parameters. A simulation study is performed to evaluate the distribution performance.

2. Mathematical Properties of OLG family

2.1. Useful representations pdf and cdf

We can expand the cdf of OLG family using Eq.(5) as follows: Using binomial series expansion we get:

$$\left(1\frac{G(x,\xi)\log(1-G(x,\xi))}{\gamma}\right)^{-\alpha}=\sum_{k=0}^{\infty}\frac{\Gamma(\alpha+k)\gamma^{-k}}{k!\Gamma(\alpha)}G(x,\xi)^{k}\left(\log(1-G(x,\xi))\right)^{k}$$

Also by using logarithm expansion of $(\log(1-G(x,\xi)))^{k}$ by the form:

$$\left[\log(1-G(x,\xi))\right]^{k}=\sum_{i=0}^{\infty}(-1)^{i}d_{k,i}\left(G(x,\xi)\right)^{i+k}$$

Where $d_{k,i}=i^{-1}\sum_{m=1}^{i}\frac{[m(k+1)-i]}{m+1}$ for $i\geq 0$ and $d_{k,0}=1$

$$F_{OL}(x,\alpha,\gamma,\xi)=1-\sum_{k=i=0}^{\infty}\frac{\Gamma(\alpha+k)\gamma^{-k}(-1)^{i}d_{k,i}}{K!\Gamma(\alpha)}G(x,\xi)^{2k+i}$$

$$G(x,\xi)^{2k+i}=\left[1-(1-G(x,\xi))\right]^{2k+i}$$
$$=\sum_{j=0}^{\infty}(-1)^{j}\binom{2k+1}{j}\left[\,1-G(x,\xi)\right]^{j}$$

Again by using binomial series expansion we get:

$$G(x,\xi)^{2k+i}=\sum_{r=j=0}^{\infty}(-1)^{j+r}\binom{2k+1}{j}\binom{j}{r}\left[G(x,\xi)\right]^{r}$$

Finally we get the expansion cdf for OLG family by the form:

$$F_{OLG}(x,\alpha,\gamma,\xi)=1-\sum_{k=i=r=j=0}^{\infty}\frac{\Gamma(\alpha+k)\gamma^{-k}(-1)^{i+j+r}d_{k,i}}{K!\Gamma(\alpha)}\binom{2k+1}{j}\binom{j}{r}\left[G(x,\xi)\right]^{r}\tag{7}$$

Or

$$F_{OLG}(x,\alpha,\gamma,\xi)=1-\Phi_{k,i,r,j}\left[G(x,\xi)\right]^{r}\tag{8}$$

Where

$$\Phi_{k,i,r,j}=\sum_{k=i=r=j=0}^{\infty}\frac{\Gamma(\alpha+k)\gamma^{-k}(-1)^{i+j+r}d_{k,i}}{K!\Gamma(\alpha)}\binom{2k+1}{j}\binom{j}{r}$$

We can expand the pdf of OLG family using Eq.(6) as follows: Using binomial series expansion we get:

$$f_{OL}(x,\alpha,\gamma,\xi)=\sum_{k=0}^{\infty}\frac{\alpha\Gamma(\alpha+1+k)\gamma^{-(k+1)}}{k!\Gamma(\alpha+1)}\frac{G(x,\xi)^{k+1}}{1-G(x,\xi)}\left(\log(1-G(x,\xi))\right)^{k}g(x,\xi)$$
$$-\sum_{k=0}^{\infty}\frac{\alpha\Gamma(\alpha+1+k)\gamma^{-(k+1)}}{k!\Gamma(\alpha+1)}G(x,\xi)^{k}\left(\log(1-G(x,\xi))\right)^{k+1}g(x,\xi)$$

Also by using logarithm expansion of $(\log(1-G(x,\xi)))^{k}$, and $(\log(1-G(x,\xi)))^{k+1}$ we get:

$$f_{OL}(x,\alpha,\gamma,\xi)=\sum_{k=i=0}^{\infty}\frac{(-1)^{i}d_{k,i}\alpha\,\Gamma(\alpha+1+k)\gamma^{-(k+1)}}{k!\Gamma(\alpha+1)}\frac{G(x,\xi)^{2k+i}}{1-G(x,\xi)}g(x,\xi)$$

$$-\sum_{k=l=0}^{\infty}\frac{\alpha\left(-1\right)^{i}d_{k+1,i}\Gamma\left(\alpha+1+k\right)\gamma^{-(k+1)}}{k!\Gamma\left(\alpha+1\right)}\left(G\left(x,\xi\right)\right)^{l+2k+1}g\left(x,\xi\right)$$

Where $d_{k,i}=i^{-1}\sum_{m=1}^{i}\frac{[m(k+1)-i]}{m+1}$ for $i\geq 0$ and $d_{k,0}=1$

And where $d_{k+i,l}=l^{-1}\sum_{m=1}^{i}\frac{[m(k+2)-l]}{m+1}$for $l\geq 0$ and $d_{k+1,0}=1$

By expansion $G\left(x,\xi\right)^{2k+i}, G\left(x,\xi\right)^{2k+l+1}$, and $\frac{1}{1-G(x,\xi)}$, respectively by the forms:

$$G\left(x,\xi\right)^{2k+i}=\sum_{r=j=0}^{\infty}\left(-1\right)^{j+r}\binom{2k+i}{j}\binom{j}{r}\left[G\left(x,\xi\right)\right]^{r}$$

$$G\left(x,\xi\right)^{2k+l+1}=\sum_{s=n=0}^{\infty}\left(-1\right)^{s+n}\binom{2k+l+1}{s}\binom{s}{n}\left[G\left(x,\xi\right)\right]^{n}$$

$$\frac{1}{1-G\left(x,\xi\right)}=\sum_{z=0}^{\infty}\left(-1\right)^{z}\left[G\left(x,\xi\right)\right]^{z}$$

Finally we get the expansion pdf for OLG family by the form:

$$f_{OLG}\left(x,\alpha,\gamma,\xi\right)=\sum_{k=i=r=j=z=0}^{\infty}\frac{\left(-1\right)^{i+r+j+z}d_{k,i}\alpha\Gamma\left(\alpha+1+k\right)\gamma^{-(k+1)}}{k!\Gamma\left(\alpha+1\right)}\binom{2k+i}{j}\binom{j}{r}\left[G\left(x,\xi\right)\right]^{r+z}g\left(x,\xi\right)$$
$$-\sum_{k=l=s=n=0}^{\infty}\frac{\alpha\left(-1\right)^{i+s+n}d_{k+1,i}\Gamma\left(\alpha+1+k\right)\gamma^{-(k+1)}}{k!\Gamma\left(\alpha+1\right)}\binom{2k+l+1}{s}\binom{s}{n}\left[G\left(x,\xi\right)\right]^{n}g\left(x,\xi\right) \tag{9}$$

Or

$$f_{OLG}\left(x,\alpha,\gamma,\xi\right)=\Phi_{k,i,r,j,z}\left[G\left(x,\xi\right)\right]^{r+z}g\left(x,\xi\right)-\psi_{k,l,s,n}\left[G\left(x,\xi\right)\right]^{n}g\left(x,\xi\right) \tag{10}$$

Where

$$\Phi_{k,i,r,j,z}=\sum_{k=i=r=j=z=0}^{\infty}\frac{\left(-1\right)^{i+r+j+z}d_{k,i}\alpha\ \Gamma\left(\alpha+1+k\right)\gamma^{-(k+1)}}{k!\Gamma\left(\alpha+1\right)}\binom{2k+i}{j}\binom{j}{r}$$

$$\psi_{k,l,s,n}=\sum_{k=l=s=n=0}^{\infty}\frac{\alpha\left(-1\right)^{i+s+n}d_{k+1,i}\Gamma\left(\alpha+1+k\right)\gamma^{-(k+1)}}{k!\Gamma\left(\alpha+1\right)}\binom{2k+l+1}{s}\binom{s}{n}$$

2.2. Quantile function

The quantile function $Q(u)$ is obtained from the relation:

$$Q\left(u\right)=F^{-1}(u)$$

Where $Q\left(u\right)$ is the quantity function $F_{OLG}\left(x,\alpha,\gamma,\xi\right)$ for each $u\in(0,1)$. We find the quantity function for the OLG family by assuming: $m=G\left(x,\xi\right)$

$$u=1-\left(\frac{\gamma}{\gamma-m.\log\left(1-m\right)}\right)^{\alpha}\Longrightarrow\frac{\gamma}{\gamma-m.\log\left(1-m\right)}=\left(1-u\right)^{\frac{1}{\alpha}}$$

$$\Longrightarrow\frac{\gamma}{\left(1-u\right)^{\frac{1}{\alpha}}}=\gamma-m.\ \log\left(1-m\right)\Longrightarrow m.\ \log\left(1-m\right)=\gamma-\frac{\gamma}{\left(1-u\right)^{\frac{1}{\alpha}}}$$

Putting

$$c=\gamma-\frac{\gamma}{\left(1-u\right)^{\frac{1}{\alpha}}}\Longrightarrow m.\log\left(1-m\right)=c\Longrightarrow\log\left(1-m\right)=\frac{c}{m}\ldots(*)$$

Let

$$\theta = \frac{c}{m} \Longrightarrow m = \frac{c}{\theta} \Longrightarrow 1 - m = 1 - \frac{c}{\theta} = \frac{\theta - c}{\theta}$$

Substituting into the equation $(*)$ we get:

$$\log\left(\frac{\theta - c}{\theta}\right) = \theta \Longrightarrow \frac{\theta - c}{\theta} = e^{\theta} \Longrightarrow e^{-\theta}\frac{\theta - c}{\theta} = 1$$

Theorem 2.1. *The solution (s) of the equation $e^{ax}.\frac{x-p}{x-q} = b$ are $x = p + \frac{1}{a}.W\_be^{-ap}\left(abe^{-ap}T\right), W\begin{pmatrix} q; & b \\ p; & \end{pmatrix} = p + W\_be^{-p}\left(be^{-p}T\right)$, hence $T = p - q$*

From Theorem (2.1) Put$a = -1, p = c, q = 0, b = 1$, we get:

$$\theta = c + W_{-1}\left(ce^{-c}\right) \Longrightarrow \frac{c}{m} = c + W_{-1}\left(ce^{-c}\right) \Longrightarrow m = \frac{c}{c + W_{-1}\left(ce^{-c}\right)}$$

$$\therefore G(x,\xi) = \frac{\gamma - \frac{\gamma}{(1-u)^{\frac{1}{\alpha}}}}{\gamma - \frac{\gamma}{(1-u)^{\frac{1}{\alpha}}} + W_{-1}\left(\left(\gamma - \frac{\gamma}{(1-u)^{\frac{1}{\alpha}}}\right)e^{-\left(\gamma - \frac{\gamma}{(1-u)^{\frac{1}{\alpha}}}\right)}\right)}$$

$$Q_{F_{OL}(x,\alpha,\gamma,\xi)} = Q_u\left(\frac{\gamma - \frac{\gamma}{(1-u)^{\frac{1}{\alpha}}}}{\gamma - \frac{\gamma}{(1-u)^{\frac{1}{\alpha}}} + W_{-1}\left(\left(\gamma - \frac{\gamma}{(1-u)^{\frac{1}{\alpha}}}\right)e^{-\left(\gamma - \frac{\gamma}{(1-u)^{\frac{1}{\alpha}}}\right)}\right)}\right) \tag{11}$$

2.3. Moments

Let x be a random variable with pdf in Eq.(10). Then the n^{th} moment of the OLG family distribution is given by:

$$\mu_n = E\left(x^n\right)_{OLoG} = \int_0^{\infty} x^n f_{OLE}\left(x,\alpha,\gamma,\xi\right)_{OLE} dx$$

$$\mu_n = \Phi_{k,i,r,j,z}\int_0^{\infty} x^n\left[G(x,\xi)\right]^{r+z} g(x,\xi)\,dx - \Phi_{k,i,r,j,z}\int_0^{\infty} x^n\left[G(x,\xi)\right]^{n} g(x,\xi)\,dx \tag{12}$$

2.4. Moment generating function

The moment generating function (mgf) is given by:

$$M_x(y)_{OLG} = E\left(e^{yx}\right) = \int_{-\infty}^{\infty} e^{yx} f_{OLG}\left(x,\alpha,\gamma,\xi\right)dx$$

Used series expansion for e^{yx}

$$M_x(y)_{OLG} = \sum_{n=0}^{\infty}\frac{y^n}{n!}E\left(x^n\right) = \sum_{n=0}^{\infty}\frac{y^n}{n!}[\mu_n]$$

From Eq.(12) we get:

$$M_x(y)_{OLG} = \sum_{n=0}^{\infty}\frac{y^n}{n!}\left[\Phi_{k,i,r,j,z}\int_0^{\infty} x^n\left[G(x,\xi)\right]^{r+z} g(x,\xi)\,dx - \Phi_{k,i,r,j,z}\int_0^{\infty} x^n\left[G(x,\xi)\right]^{n} g(x,\xi)\,dx\right] \tag{13}$$

2.5. Rényi entropy

The Rényi entropy for the OLG family distribution can be obtained [11]:

$$I_R\left(c\right)_{OLG} = \frac{1}{1-c} log \int_0^{\infty} f\left(x\right)^c dx$$

Then from Eq.(10) we get:

$$I_R\left(c\right)_{OLG} = \frac{1}{1-c} \log \left[\int_0^{\infty} \left(\Phi_{k,i,r,j,z} \left[G\left(x,\xi\right)\right]^{r+z} g\left(x,\xi\right) dx - \psi_{k,l,s,n} \left[G\left(x,\xi\right)\right]^{n} g\left(x,\xi\right) \right)^c dx \right] \tag{14}$$

2.6. Order statistics

The pdf of the j^{th} order statistic for a random sample of size n from a distribution function $F_{OLG}\left(x,\alpha,\gamma,\xi\right)$ and an associated pdf $f_{OLG}\left(x,\alpha,\gamma,\xi\right)$ are given by:

$$f_{j:n}\left(x\right) = \sum_{r=0}^{n-j} k\left(-1\right)^r \binom{n-j}{r} \left[F_{OLG}\left(x,\alpha,\gamma,\xi\right)\right]^{j+r-1} f_{OLG}\left(x,\alpha,\gamma,\xi\right) \tag{15}$$

3. The OLE distribution

If we take the Exponential (E) distribution as a baseline model, the cdf and pdf of E is $G\left(t\right) = 1 - e^{-\lambda x}$ and $g\left(t\right) = \lambda e^{-\lambda x}$, respectively and substitute in Eq.(5), and Eq.(6) to have the cdf and pdf of odd Lomax exponential distribution (OLE), respectively, by:

$$F_{OLE}\left(x,\alpha,\gamma,\lambda\right) = 1 - \left(1 - \frac{\left(1-e^{-\lambda x}\right).\log\left(e^{-\lambda x}\right)}{\gamma}\right)^{-\alpha} ,x > 0,\ \alpha,\gamma,\lambda > 0 \tag{16}$$

$$f_{OLE}\left(x,\alpha,\gamma,\xi\right) = \frac{\alpha\lambda}{\gamma} \left(1 - \frac{\left(1-e^{-\lambda x}\right).\ \log\left(e^{-\lambda x}\right)}{\gamma}\right)^{-(\alpha+1)} . \left[1 - e^{-\lambda x} - e^{-\lambda x} \log\left(e^{-\lambda x}\right)\right] \tag{17}$$

The Survival function can be getting from following equation:

$$S\left(x\right)_{OLE} = 1 - F_{OLE}\left(x,\alpha,\gamma,\xi\right)$$

We get

$$S\left(x\right)_{OLE} = \left(1 + \frac{\left(1-e^{-\lambda x}\right).\log\left(e^{-\lambda x}\right)}{\gamma}\right)^{-\alpha} \tag{18}$$

While the hazard functions of the OLE distribution is obtained by Equation:

$$h\left(x\right)_{OLE} = \frac{pdf}{S\left(x\right)_{OLE}} = \frac{f_{OLE}\left(x,\alpha,\gamma,\xi\right)}{S\left(x\right)_{OLE}}$$

$$h\left(x\right)_{OLE} = \frac{\alpha\lambda.\left[1 - e^{-\lambda x} - e^{-\lambda x}\log\left(e^{-\lambda x}\right)\right]}{\gamma + \left(1-e^{-\lambda x}\right).\log\left(e^{-\lambda x}\right)} \tag{19}$$

Then the Figure 1 represents pdf, and hazard function of OLE distribution, respectively

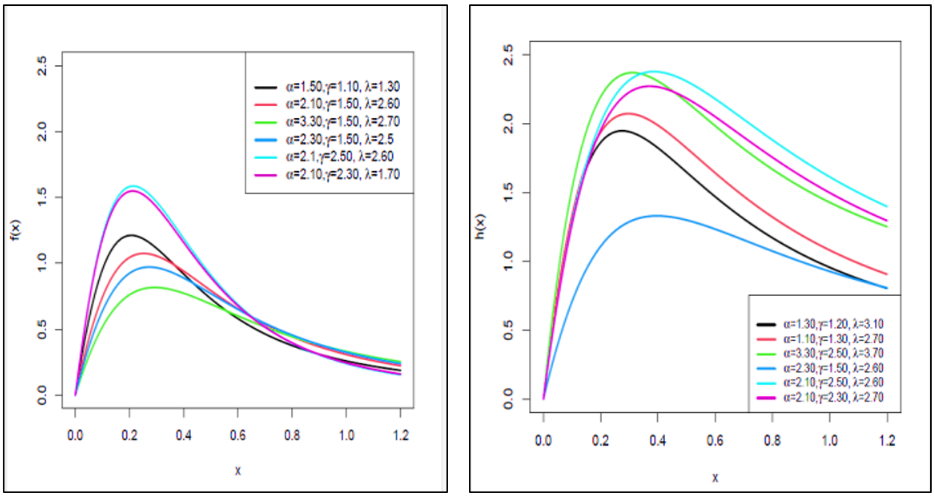

Figure 1: pdf, and hazard function OLE distribution

4. Mathematical properties

4.1. Useful representations pdf and cdf of OLE distribution

We can expand the cdf of OLE distribution using Eq.(8) as follows: Using binomial series expansion, and logarithm expansion we get [12, 13, 14]:

$$F_{OLE}(x,\alpha,\gamma,\lambda) = 1 - \sum_{k=i=r=j=p=0}^{\infty} \frac{\Gamma(\alpha+k)\,\gamma^{-k}\,(-1)^{i+j+r+p}\,d_{k,i}}{K!\Gamma(\alpha)} \binom{2k+1}{j}\binom{j}{r}\binom{r}{p} e^{-p\lambda x}$$

$$F_{OLE}(x,\alpha,\gamma,\lambda) = 1 - \Upsilon_{k,i,r,j,p}\, e^{-p\lambda x} \tag{20}$$

Where

$$\Upsilon_{k,i,r,j,p} = \sum_{k=i=r=j=p=0}^{\infty} \frac{\Gamma(\alpha+k)\,\gamma^{-k}\,(-1)^{i+j+r+p}\,d_{k,i}}{K!\Gamma(\alpha)} \binom{2k+1}{j}\binom{j}{r}\binom{r}{p}$$

We can expand the pdf of OLE distribution using Eq.(10) as follows: Using binomial series expansion, and logarithm expansion we get:

$$\begin{aligned} f_{OLE}(x,\alpha,\gamma,\xi) = & \sum_{k=i=r=j=z=q=0}^{\infty} \frac{(-1)^{i+r+j+z+q}\,d_{k,i}\alpha\;\Gamma(\alpha+1+k)\,\gamma^{-(k+1)}}{k!\Gamma(\alpha+1)} \binom{2k+i}{j}\binom{j}{r}\binom{r+z}{q} \lambda e^{-\lambda x(q+1)} \\ & - \sum_{k=l=s=n=v=0}^{\infty} \frac{\alpha\,(-1)^{i+s+n+v}\,d_{k+1,i}\Gamma(\alpha+1+k)\,\gamma^{-(k+1)}}{k!\Gamma(\alpha+1)} \binom{2k+l+1}{s}\binom{s}{n}\binom{n}{v} \lambda e^{-\lambda x(v+1)} \end{aligned}$$

Or

$$f_{OLE}(x,\alpha,\gamma,\xi) = \varpi_{k,i,r,j,z,q}\, e^{-\lambda x(q+1)} - \vartheta_{k,l,s,n,v}\, e^{-\lambda x(v+1)} \tag{21}$$

Where

$$\varpi_{k,i,r,j,z,q} = \sum_{k=i=r=j=z=q=0}^{\infty} \frac{(-1)^{i+r+j+z+q}\,d_{k,i}\alpha\;\Gamma(\alpha+1+k)\,\gamma^{-(k+1)}}{k!\Gamma(\alpha+1)} \binom{2k+i}{j}\binom{j}{r}\binom{r+z}{q} \lambda$$

And

$$\vartheta_{k,l,s,n,v} = \sum_{k=l=s=n=v=0}^{\infty} \frac{\alpha\,(-1)^{i+s+n+v}\,d_{k+1,i}\Gamma(\alpha+1+k)\,\gamma^{-(k+1)}}{k!\Gamma(\alpha+1)} \binom{2k+l+1}{s}\binom{s}{n}\binom{n}{v} \lambda$$

4.2. Quantile function of OLE distribution

From Eq.(11) we can get the Quantile function of OLE distribution by form:

$$Q_{F_{OLE}(x,\alpha,\gamma,\xi)} = Q_u\left(-\frac{1}{\lambda}\ln\left(1-\frac{\gamma-\frac{\gamma}{(1-u)^{\frac{1}{\alpha}}}}{\gamma-\frac{\gamma}{(1-u)^{\frac{1}{\alpha}}}+W_{-1}\left(\left(\gamma-\frac{\gamma}{(1-u)^{\frac{1}{\alpha}}}\right)e^{-\left(\gamma-\frac{\gamma}{(1-u)^{\frac{1}{\alpha}}}\right)}\right)}\right)\right) \tag{22}$$

Table 1: explains the quantiles for selected parameter values of the OLE distribution.

	(α,γ,λ)				
u	(3.4, 2.7, 1.3)	(2.6, 4.3, 2.7)	(2.5, 3.3, 1.6)	(2.8, 2, 0.8)	(3.8, 2.4, 1.2)
0.1	0.2421	0.1749	0.2605	0.3722	0.2317
0.2	0.3694	0.2734	0.4043	0.5680	0.3315
0.3	0.4886	0.3704	0.5441	0.7516	0.4623
0.4	0.6130	0.4773	0.6961	0.9439	0.5764
0.5	0.7526	0.6047	0.8748	1.1608	0.7026
0.6	0.9206	0.7693	1.1020	1.4237	0.8519
0.7	1.1414	1.0046	1.4221	1.7724	1.0438
0.8	1.4742	1.3971	1.9499	2.3064	1.3249
0.9	2.1434	2.2735	3.1390	3.4122	1.8649

4.3. Moments

Let x be a random variable with pdf in Eq.(21). Then the n^{th} moment by Eq.(12), then the moment of the OLE distribution is given by [15]:

$$\mu_n = E\left(x^n\right)_{OLE} = \varpi_{k,i,r,j,z,q}\int_0^\infty x^n e^{-\lambda x(q+1)}dx + \vartheta_{k,l,s,n,v}\int_0^\infty x^n e^{-\lambda x(v+1)}\,dx$$

Let

$$y=\lambda x\left(q+1\right) \Longrightarrow x=\frac{y}{\lambda\left(q+1\right)} \Longrightarrow dx=\frac{dy}{\lambda\left(q+1\right)}$$

And let

$$r=\lambda x\left(v+1\right) \Longrightarrow x=\frac{r}{\lambda\left(v+1\right)} \Longrightarrow dx=\frac{dr}{\lambda\left(v+1\right)}$$

$$\mu_n = \varpi_{k,i,r,j,z,q}\left(\frac{1}{\lambda\left(q+1\right)}\right)^{n+1}\int_0^\infty y^n e^{-y}dy + \vartheta_{k,l,s,n,v}\left(\frac{1}{\lambda\left(q+1\right)}\right)^{n+1}\int_0^\infty r^n e^{-r}\,dr$$

$$\mu_n = \varpi_{k,i,r,j,z,q}\left(\frac{1}{\lambda\left(q+1\right)}\right)^{n+1}\Gamma\left(n+1\right) + \vartheta_{k,l,s,n,v}\left(\frac{1}{\lambda\left(q+1\right)}\right)^{n+1}\Gamma\left(n+1\right)$$

$$\mu_n = \Gamma\left(n+1\right)\left[\varpi_{k,i,r,j,z,q}\left(\frac{1}{\lambda\left(q+1\right)}\right)^{n+1} + \vartheta_{k,l,s,n,v}\left(\frac{1}{\lambda\left(q+1\right)}\right)^{n+1}\right] \tag{23}$$

The variance of the OLE distribution is obtained by the following formula ($\sigma^2 = \mu_2 - \mu_1^2$). The skewness ($SK$) and kurtosis ($KU$) are defined by

$$SK = \frac{\mu_3}{\mu_2^{\left(\frac{3}{2}\right)}} = \frac{6\left[\varpi_{k,i,r,j,z,q}\left(\frac{1}{\lambda(q+1)}\right)^4 + \vartheta_{k,l,s,n,v}\left(\frac{1}{\lambda(q+1)}\right)^4\right]}{\left[2\left[\varpi_{k,i,r,j,z,q}\left(\frac{1}{\lambda(q+1)}\right)^3 + \vartheta_{k,l,s,n,v}\left(\frac{1}{\lambda(q+1)}\right)^3\right]\right]^{\frac{3}{2}}} \tag{24}$$

$$KU = \frac{\mu_4}{\mu_2^{\left(\frac{2}{2}\right)}} = \frac{6\left[\varpi_{k,i,r,j,z,q}\left(\frac{1}{\lambda(q+1)}\right)^5 + \vartheta_{k,l,s,n,v}\left(\frac{1}{\lambda(q+1)}\right)^5\right]}{\left[\varpi_{k,i,r,j,z,q}\left(\frac{1}{\lambda(q+1)}\right)^3 + \vartheta_{k,l,s,n,v}\left(\frac{1}{\lambda(q+1)}\right)^3\right]^2} \tag{25}$$

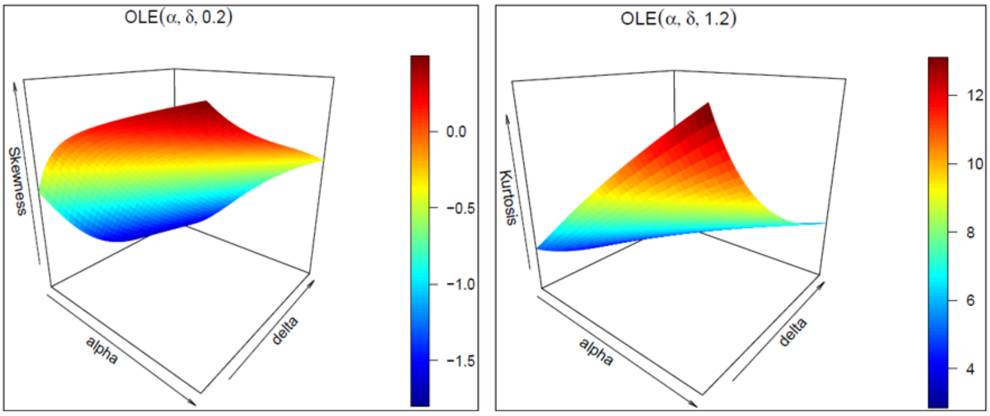

Figure 2: Displays 3D plots of skewness and kurtosis of the OLE distribution

4.4. Moment generating function

The moment generating function (mgf) of OLE distribution from Eq.(13), and from Eq.(23) getting:

$$M_x(y)_{OLE} = \sum_{n=0}^{\infty} \frac{y^n}{n!}\left[\Gamma(n+1)\left(\varpi_{k,i,r,j,z,q}\left(\frac{1}{\lambda(q+1)}\right)^{n+1} + \vartheta_{k,l,s,n,v}\left(\frac{1}{\lambda(q+1)}\right)^{n+1}\right)\right] \tag{26}$$

4.5. Rényi entropy

The Rényi entropy for the OLE distribution can be used Eq.(14), and using binomial series expansion obtained: Then

$$I_R(c)_{OLE} = \frac{1}{1-c}\log\left[\frac{\sum_{m=0}^{c}(-1)^m\binom{c}{m}\varpi_{k,i,r,j,z,q}.\vartheta_{k,l,s,n,v}}{\lambda(qm+vc-vm+c)}\right] \tag{27}$$

4.6. Order statistics

The pdf of the j^{th} order statistic for a random sample of size n from a distribution function $F_{OLE}(x,\alpha,\gamma,\xi)$ and an associated pdf $f_{OLE}(x,\alpha,\gamma,\xi)$ are given by [16, 17]:

$$f_{j:n}(x) = \sum_{r=0}^{n-j} k(-1)^r\binom{n-j}{r}\left[F_{OLE}(x,\alpha,\gamma,\xi)\right]^{j+r-1} f_{OLE}(x,\alpha,\gamma,\xi) \tag{28}$$

Where $F(x)_{GoNH}$ cdf of OLE-distribution and $f(x)_{GoNH}$ is cdf of OLE-distribution. However, the following is the pdf of the $j-th$ order statistics for a random sample of size n drawn from the OLE-distribution:

$$\begin{aligned} f_{j:n}(x) = &\sum_{r=0}^{n-j} k(-1)^r\binom{n-j}{r}\left[1-\left(1-\frac{(1-e^{-\lambda x}).\log(e^{-\lambda x})}{\gamma}\right)^{-\alpha}\right]^{j+r-1} \\ &* \left[\frac{\alpha\lambda}{\gamma}\left(1-\frac{(1-e^{-\lambda x}).\ \log(e^{-\lambda x})}{\gamma}\right)^{-(\alpha+1)}\left[1-e^{-\lambda x}-e^{-\lambda x}\log\left(e^{-\lambda x}\right)\right]\right] \end{aligned} \tag{29}$$

So, the $f_{j:n}(x)$ of minimum order statistics is obtained by substituting $j=1$ in Eq.(29) to have:

$$f_{1:n}(x)=\sum_{r=0}^{n-1}k(-1)^r\binom{n-1}{r}\left[1-\left(1+\frac{\left(1-e^{-\lambda x}\right).\log\left(e^{-\lambda x}\right)}{\gamma}\right)^{-\alpha}\right]^r$$
$$*\left[\frac{\alpha\lambda}{\gamma}\left(1-\frac{\left(1-e^{-\lambda x}\right).\ \log\left(e^{-\lambda x}\right)}{\gamma}\right)^{-(\alpha+1)}\left[1-e^{-\lambda x}-e^{-\lambda x}\log\left(e^{-\lambda x}\right)\right]\right] \tag{30}$$

While the corresponding $f_{j:n}(x)$ of maximum order statistics is obtained by making the substitution of $j=n$ in Eq.(29) as:

$$f_{n:n}(x)=\sum_{r=0}^{n-j}k(-1)^r\binom{n-j}{r}\left[1-\left(1-\frac{\left(1-e^{-\lambda x}\right).\log\left(e^{-\lambda x}\right)}{\gamma}\right)^{-\alpha}\right]^{n+r-1}$$
$$*\left[\frac{\alpha\lambda}{\gamma}\left(1-\frac{\left(1-e^{-\lambda x}\right).\ \log\left(e^{-\lambda x}\right)}{\gamma}\right)^{-(\alpha+1)}\left[1-e^{-\lambda x}-e^{-\lambda x}\log\left(e^{-\lambda x}\right)\right]\right] \tag{31}$$

5. Estimation

The parameters of the OLE distribution are estimated using the maximum likelihood estimation approach. The log-likelihood function is derived for a random sample $x_1, x_2, \ldots, x_n$ distributed in accordance with the pdf of the OLE distribution.

$$L(\ \Theta, x)=\prod_{i=1}^{n}f_{OLE}(x_i,\alpha,\gamma,\xi)$$
$$L(\ \Theta, x)=\prod_{i=1}^{n}\frac{\alpha\lambda}{\gamma}\left(1-\frac{\left(1-e^{-\lambda x_i}\right).\ \log\left(e^{-\lambda x_i}\right)}{\gamma}\right)^{-(\alpha+1)}.\left[1-e^{-\lambda x}-e^{-\lambda x}\log\left(e^{-\lambda x}\right)\right]$$

The log-likelihood function L is obtained as:

$$L=n\log(\alpha)+n\log(\lambda)-n\log(\gamma)-(\alpha+1)\sum_{i=1}^{n}\log\left(1-\frac{\left(1-e^{-\lambda x_i}\right).\ \log\left(e^{-\lambda x_i}\right)}{\gamma}\right)$$
$$+\log\left(1-e^{-\lambda x}-e^{-\lambda x}\log\left(e^{-\lambda x}\right)\right) \tag{32}$$
$$\frac{\partial L}{\partial\alpha}=\frac{n}{\alpha}-\sum_{i=1}^{n}log\left(1-\frac{\left(1-e^{-\lambda x_i}\right).\ \log\left(e^{-\lambda x_i}\right)}{\gamma}\right) \tag{33}$$
$$\frac{\partial L}{\partial\lambda}=\frac{n}{\lambda}+(\alpha+1)\sum_{i=1}^{n}\frac{x_i.\left(1-e^{-\lambda x_i}\right)-x_ie^{-\lambda x_i}\ .\log\left(e^{-\lambda x_i}\right)}{\gamma-\left(1-e^{-\lambda x_i}\right).\ \log\left(e^{-\lambda x_i}\right)}+\sum_{i=1}^{n}\frac{x_ie^{-\lambda x_i}\ .\log\left(e^{-\lambda x_i}\right)}{1-e^{-\lambda x_i}+\log e^{-\lambda x_i}} \tag{34}$$
$$\frac{\partial L}{\partial\gamma}=-\frac{n}{\gamma}-(\alpha+1)\sum_{i=1}^{n}\frac{\left(1-e^{-\lambda x_i}\right).\ \log\left(e^{-\lambda x_i}\right)}{\gamma^2-\gamma\left(1-e^{-\lambda x_i}\right).\ \log\left(e^{-\lambda x_i}\right)} \tag{35}$$

The solution of the non-linear equations of $\frac{\partial L}{\partial\alpha}=0, \frac{\partial L}{\partial\lambda}=0$, and $\frac{\partial L}{\partial\gamma}=0$ results to the ML estimates of parameters α, λ, γ respectively. The solution could not be obtained analytically except by numerical methods using software like R, MAPLE, and SAS and so on.

6. Simulation study

The performance of maximum likelihood estimators (MLEs) for the OLE distribution is evaluated through a Monte Carlo simulation study using the R package. The sample sizes considered in the study are n = 100, 150, and 175. We generate N = 1000 samples for the true parameter values listed in Tables 2 and 3. The resulting MLEs for the model parameters are averaged to obtain the mean values, and the corresponding bias and root mean squared errors (RMSEs) are calculated. The bias and RMSE for a specific estimated parameter, denoted as $\hat{\gamma}$, are given by:

$$Abias\left(\hat{\gamma}\right)=\frac{\sum_{i=1}^{N}\hat{\gamma}_i}{N}-\gamma,\ \text{and}\ RMSE\left(\hat{\gamma}\right)=\sqrt{\frac{\sum_{i=1}^{N}\left(\hat{\gamma}_i-\gamma\right)^2}{N}}.$$

The result in Tables 2 and Table 3 are shown, all estimators appear to have consistency. That is, as the sample size increases, the average parameter estimates become closer to the true parameter values. In addition, the mean square errors (MSEs) decrease as the sample size increases.

Table 2: Outcomes of Monte Carlo simulations conducted for the OLE distribution

	$(\alpha=1.1,\gamma=1.1,\gimel=0.8)$				$(\alpha=1.1,\gamma=0.4,\gimel=0.8)$		
parameter	Sample Size	Mean	RMSE	Abias	Mean	RMSE	Abias
	100	1.9490	1.0644	0.8490	2.2238	2.9803	1.123
α	150	1.8901	0.9880	0.7901	1.8775	1.7122	0.7775
	175	1.7114	0.6481	0.6114	1.7999	0.8956	0.6999
	100	3.0276	2.9443	1.9276	2.9709	8.6495	2.5709
γ	150	2.8421	2.7471	1.7421	2.1456	6.3173	1.7456
	175	2.1900	1.2308	1.0900	1.6423	1.9048	1.2423
	100	0.9116	0.2545	0.1116	1.2229	0.7054	0.4229
$\gimel$	150	0.8625	0.2367	0.0625	1.1825	0.5930	0.3825
	175	0.8537	0.1674	0.0537	1.1411	0.5423	0.3411
	$(\alpha=1.6,\gamma=1.6,\gimel=0.6)$				$(\alpha=1.1,\gamma=1.1,\gimel=0.6)$		
parameter	Sample Size	Mean	RMSE	Abias	Mean	RMSE	Abias
	100	3.2222	4.2954	1.6229	2.2190	1.3371	1.1190
α	150	2.5353	1.6619	0.9353	1.8924	0.9233	0.7924
	175	2.5173	1.2661	0.9173	1.7179	0.6817	0.6179
	100	6.8865	4.8809	5.2865	4.0533	3.8907	2.9533
γ	150	3.8616	4.5153	2.2616	3.1353	2.7674	2.0353
	175	3.7498	3.4711	2.1498	2.0917	1.1547	0.9917
	100	0.7146	0.2989	0.1146	0.7201	0.2403	0.1201
$\gimel$	150	0.6675	0.1968	0.0675	0.6853	0.2360	0.0853
	175	0.6641	0.1893	0.0641	0.5994	0.0545	-0.0005

Table 3: Outcomes of Monte Carlo simulations conducted for the OLE distribution

		$(\alpha = 1.4, \gamma = 1.4, \gimel = 0.4)$			$(\alpha = 1.4, \gamma = 1.5, \gimel = 0.4)$		
parameter	Sample Size	Mean	RMSE	Abias	Mean	RMSE	Abias
α	100	2.6731	2.2244	1.2731	2.6729	2.1262	1.2729
	150	2.2560	1.7569	0.8560	2.2917	1.8085	0.8917
	175	2.1294	0.9831	0.7294	2.1400	0.9978	0.7400
γ	100	5.2875	8.8442	3.8875	4.9390	6.7219	3.4390
	150	4.0049	8.2386	2.6049	4.1960	6.1865	2.6960
	175	2.8272	2.2470	1.4272	3.0916	2.4728	1.5916
$\gimel$	100	0.4475	0.1797	0.0475	0.4317	0.1447	0.0317
	150	0.4197	0.1401	0.0197	0.4191	0.1357	0.0191
	175	0.4081	0.0781	0.0081	0.4122	0.0835	0.0122
		$(\alpha = 1.4, \gamma = 1.5, \gimel = 0.5)$			$(\alpha = 1.2, \gamma = 0.9, \gimel = 0.5)$		
parameter	Sample Size	Mean	RMSE	Abias	Mean	RMSE	Abias
α	100	2.6419	1.9974	1.2419	2.2406	1.5198	1.0406
	150	2.2105	1.7122	0.8105	1.9137	1.2305	0.7137
	175	2.0760	0.9413	0.6760	1.8981	0.8521	0.6981
γ	100	4.9974	6.9593	3.4974	3.5270	5.0073	2.6270
	150	4.0475	5.7905	2.5475	2.7748	4.8799	1.8748
	175	2.9642	2.2374	1.4642	2.0650	1.6523	1.1650
$\gimel$	100	0.5452	0.2016	0.0452	0.6173	0.2697	0.1173
	150	0.5246	0.1700	0.0246	0.5718	0.2231	0.0718
	175	0.5243	0.1016	0.0243	0.5438	0.1346	0.0438

7. Application

To demonstrate the effectiveness of the OLE distribution in fitting data, we present a practical application on two data sets for the purpose of demonstrating the advantages of OLE and the extent to which it fits the data. Table 4 provides a comparison between OLE and different distributions for the data used.

This comparison is made using eight measures, which are the Kolmogorov-Smirnov statistic (KS), the Anderson-Darling statistic (A), the Cramér-von Mises statistic (W), the information criteria HQIC, BIC, AIC, and CAIC, and the p-value corresponding to the KS test. These are widely used measures of goodness of fit.

Table 5 and Table 6 show the OLE distribution with the smallest values of AIC, AICC, and BIC compared to the corresponding values for non-overlapping distributions. Moreover, goodness-of-fit statistics A, W, KS tests and p-value indicate that the OLE distribution is the best fit for data I and data II.

Table 4: Comparative distributions

Distribution	CDF
Beta Exponential distribution (BeE) [18]	$p\gimel(1-e^{-\gimel x}, \alpha, \gamma)$
Kumaraswamy Exponential distribution (KuE) (New)	$1 - \left(1 - \left(1-e^{-\gimel x}\right)^{\alpha}\right)^{\gamma}$
Exponential Generalized Exponential distribution (EGE) (New)	$\left(1 - \left(1-e^{-\gimel x}\right)^{\alpha}\right)^{\gamma}$
Weibull Exponential distribution (WeE) [19]	$\left(1 - exp\left(-\gamma^{-\alpha}\left(-log\left(e^{-\gimel x}\right)\right)^{\alpha}\right)\right.$
Gompertz Exponential distribution (GoE) (New)	$\left(1 - \left(e^{-\gimel x}\right)^{\alpha}\right)^{\gamma} * \left(1 - exp\left(\left(\frac{\alpha}{\gamma}\right)\left(1-\left(e^{-\gimel x}\right)^{-\gamma}\right)\right)\right)$
Rayleigh Exponential distribution (RE) (New)	$e^{\frac{-\gamma}{2}\left(-\ln(1-e^{-\gimel x})\right)^2}$
Rayleigh distribution (R) [20]	$1 - exp\left(-\gimel x^2\right)$

7.1. The First Dataset I

The dataset used in this study includes the survival times of 72 guinea pigs that were infected with virulent tubercle bacilli. The survival times are measured in days. The original observation and reporting of this dataset were conducted by Bjerkedal [21].
0.1, 0.33, 0.44, 0.56, 0.59, 0.59, 0.72, 0.74, 0.92, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 1.07, 1.08, 1.08, 1.08, 1.09, 1.12, 1.13, 1.15, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 1.36, 1.39, 1.44, 1.46, 1.53, 1.59, 1.6, 1.63, 1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16, 2.22, 2.3, 2.31, 2.4, 2.45, 2.51, 2.53, 2.54, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32, 4.58 , 5.55.

Table 5: Estimates of models for data I

Dist.	MLEs	-2L	AIC	CAIC	BIC	HQIC	W	A	K-S	p-value
OLE	$\hat{\alpha}$:0.3488 $\hat{\gamma}$:3.3766 $\hat{\gimel}$:2.5631	92.90	191.81	192.16	198.64	194.53	0.0502	0.3175	0.0686	0.8860
BeE	$\hat{\alpha}$:3.3629 $\hat{\gamma}$:1.4159 $\hat{\gimel}$:0.8500	94.42	194.85	195.20	201.68	197.57	0.0832	0.5186	0.0949	0.5354
KuE	$\hat{\alpha}$:3.1070 $\hat{\gamma}$:1.5504 $\hat{\gimel}$:0.8133	94.32	194.64	194.99	201.47	197.36	0.0856	0.5257	0.0924	0.5695
EGE	$\hat{\alpha}$:1.2167 $\hat{\gamma}$:3.2351 $\hat{\gimel}$:0.8827	94.61	195.23	195.58	202.06	197.95	0.0807	0.5087	0.1105	0.3420
WeE	$\hat{\alpha}$:1.9719 $\hat{\gamma}$:0.9977 $\hat{\gimel}$:0.1657	82.49	170.98	171.75	175.65	172.59	0.1623	1.0172	0.1431	0.4704
GoE	$\hat{\alpha}$:0.7730 $\hat{\gamma}$:1.1631 $\hat{\gimel}$:0.3831	102.8	211.65	212.00	218.48	214.37	0.2938	1.7091	0.1749	0.0243
RE	$\hat{\gamma}$:1.7908 $\hat{\gimel}$:0.3598	97.30	198.61	198.79	203.17	200.43	0.1048	0.7175	0.1267	0.1977
R	$\hat{\gimel}$:0.2396	96.58	195.16	195.21	197.43	196.06	0.1760	1.0289	0.1087	0.3615

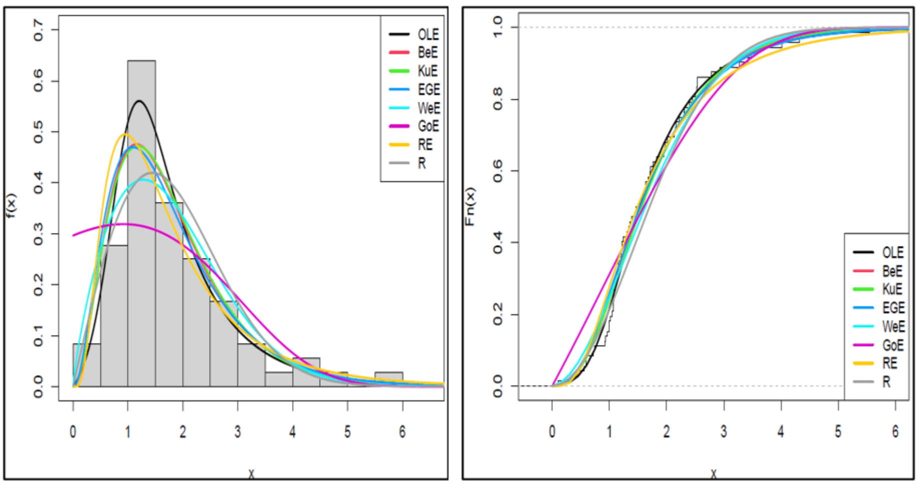

Figure 3: Fitted densities and empirical CDF for Data I

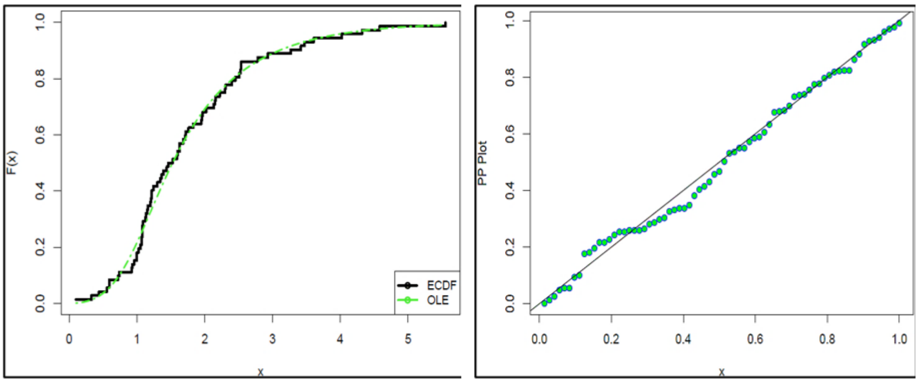

Figure 4: Empirical CDF and pp plot for data I

7.2. The Second Dataset II

The dataset contains the recorded time durations of growth hormone medication until children reach a specific target age [22].
2.15, 2.20, 2.55, 2.56, 2.63, 2.74, 2.81, 2.90, 3.05, 3.41, 3.43, 3.43, 3.84, 4.16, 4.18, 4.36, 4.42, 4.51,4.60, 4.61, 4.75, 5.03, 5.10, 5.44, 5.90, 5.96, 6.77, 7.82, 8.00, 8.16, 8.21, 8.72, 10.40, 13.20, 13.70.

Table 6: Estimates of models for data II

Dist.	MLEs	-2L	AIC	CAIC	BIC	HQIC	W	A	K-S	p-value
OLE	$\hat{\alpha}$:0.2855 $\hat{\gamma}$:6.8688 $\hat{\lambda}$:1.0943	78.95	163.91	164.68	168.58	165.52	0.0491	0.3585	0.0791	0.9806
BeE	$\hat{\alpha}$:3.6195 $\hat{\gamma}$:1.2501 $\hat{\lambda}$:0.3163	80.40	166.81	167.58	171.47	168.42	0.0861	0.5678	0.1097	0.7926
KuE	$\hat{\alpha}$:4.0029 $\hat{\gamma}$:1.2763 $\hat{\lambda}$:0.3458	79.89	165.80	166.57	170.46	167.41	0.0857	0.5654	0.1005	0.8710
EGE	$\hat{\alpha}$:1.0083 $\hat{\gamma}$:3.7409 $\hat{\lambda}$:0.3776	80.36	166.73	167.51	171.40	168.35	0.0820	0.5435	0.1134	0.7580
WeE	$\hat{\alpha}$:1.9719 $\hat{\gamma}$:0.9977 $\hat{\lambda}$:0.1657	82.49	170.98	171.75	175.65	172.59	0.1623	1.0172	0.1431	0.4704
GoE	$\hat{\alpha}$:0.4809 $\hat{\gamma}$:0.9302 $\hat{\lambda}$:0.1871	87.09	180.19	180.97	184.86	181.80	0.2516	1.5539	0.2091	0.0935
RE	$\hat{\gamma}$:1.4917 $\hat{\lambda}$:0.1033	82.42	168.86	169.23	171.97	169.93	0.0613	0.4194	0.1456	0.4476
R	$\hat{\lambda}$:0.0274	82.48	166.97	167.09	168.53	167.51	0.1643	1.0289	0.1463	0.4412

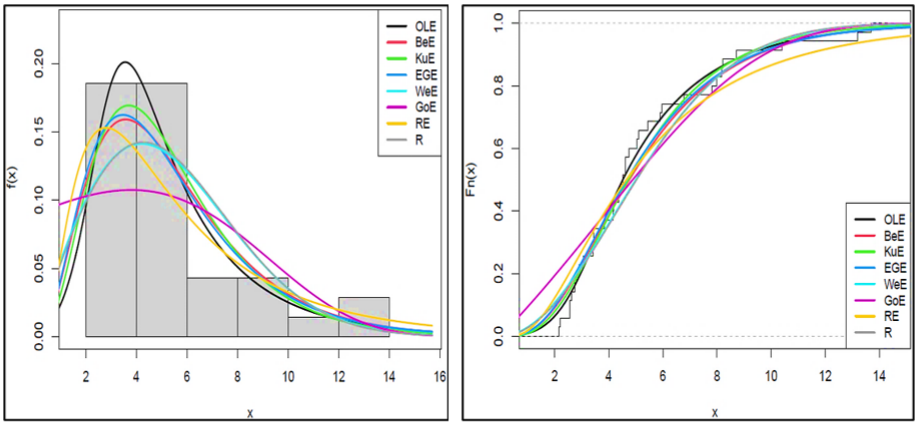

Figure 5: Fitted densities and empirical CDF for Data II

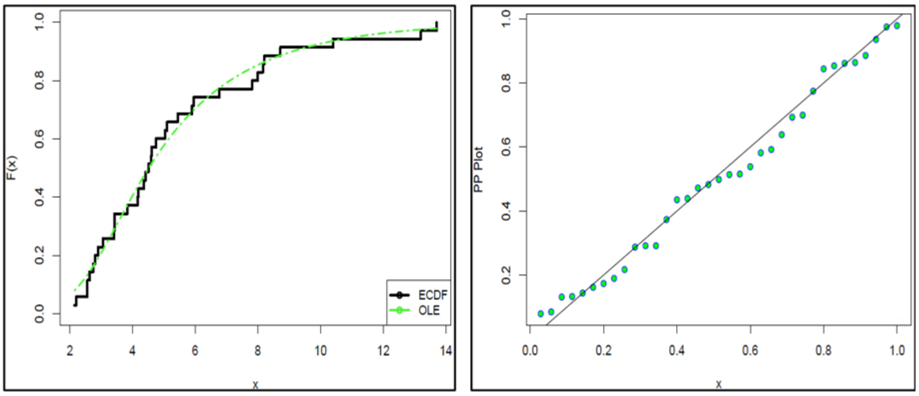

Figure 6: Empirical CDF and pp plot for data II

8. Conclusion

This study proposes the formation of a new extension to a new generalized family of continuous distributions called Odd Lomax-G (OLG) derived based on the T-X family and the Lomax distribution. It presented a set of properties for this generalized family such as quantitative function, moments, MGF, Rényi entropy, and order statistics. The study also presented the derivation of a sub-model of the exponential distribution called OLE, and a set of properties based on OLE. The performance of maximum likelihood estimators (MLEs) for the OLE distribution is evaluated through a Monte Carlo simulation study using the R package for different sample sizes, and the average MLEs were calculated. Output the model parameters to obtain mean values, and the corresponding bias and root mean square errors (RMSEs) are calculated. The bias and RMSE are given for a specific estimation parameter, and examining the results shows that all estimators are consistent. This means that as the sample size increases, the mean parameter estimates become closer to the true parameter values. In addition, the mean square errors (MSEs) decrease as the sample size increases.

Finally, to analyze the performance and superiority of the OLE model, two sets of data were used: the survival times of 72 guinea pigs infected with virulent tuberculosis bacilli and the time periods recorded for growth hormone medications until the children reach a specific target age by comparing them to some existing distributions. The comparison demonstrated the strength of the performance of the proposed model according to AIC, CAIC, BIC, and HQIC informatics criteria.

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