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Exploring the Properties, Simulation, and Applications of the Odd Burr XII Gompertz Distribution

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Abstract

This study proposes a new distribution that is more flexible than the corresponding four-parameter distributions called the Odd Burr XII Gompertz (OBXIIGo) distribution. Then various basic statistical properties of the OBXIIGo distribution are investigated and to estimate its parameters MLE is used. In addition, a Monte Carlo simulation study is conducted to evaluate the performance of parameter estimation using the MLE method, and the OBXIIGo distribution is applied to illustrate its uses on two real data sets, demonstrating its adaptability in various application fields. The results demonstrate the flexibility of the OBXIIGo distribution and its ability for auditing modeling and analysis.

Keywords: OBXIIGo distribution; Burr XII distribution; Quantile function; Ordered statistics; Moment MLE.
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1. Introduction

In recent decades, statistics has made qualitative breakthroughs in exploring new distributions. These breakthroughs focus on families of continuous distributions. The field of statistics has saw this progress. The basic models were developed with one or more shape parameters. These parameters make the models flexible and improve their capabilities. Many researchers are attracted to this trend. Statisticians find these new generators valuable because they are based on classical distributions.

This principle has led to the emergence of many families of continuous distributions. For instance, one example is the Odd Chen-G family by [1]. The Generalized Marshall-Olkin-G family, mentioned by [2], is a

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notable example. It was followed by the Weibull-G Poisson family, known for effectively modeling real data and addressing modeling challenges [3]. The Nadarajah-Haghighi family is unique and offers flexibility and strength in modeling [4]. Other families, like the generalized single exponential family, offer different distributions. These distributions can handle different amounts of data and improve modeling accuracy [5, 6]. The transformed Weibull G family is another alternative distribution. Followed by applying a set of distributions within these families. The Weibull BXII distribution [7] is one example. The Lindley BXII distribution is also unique [8]. Al sadat et al. [9] studied the Marshall-Olkin Weibull-Burr XII distribution, exploring its characteristics and applications. Gad et al. In 2019, we investigated the properties and usefulness of Burr XII-Burr XII distribution [10]. In 2018, Altun et al. studied the Zografos-Balakrishnan BXII distribution. Their research contributed to the understanding and utilization of this distribution [11]. The CDF of the new Odd Burr XII (OBXII) Family is defined using the method proposed by Alzaatreh et al. [12]. The following procedures are employed to obtain the CDF.
Let

$$W\left(F\left(x;\xi\right)\right)=\frac{F\left(x;\xi\right)}{1-F\left(x;\xi\right)}*F\left(x;\xi\right)$$

Where $W\left(F\left(x;\xi\right)\right)$ is satisfy the conditions:

1. $W\left(F\left(x;\xi\right)\right)\in\left[a,b\right],\ -\infty<a<b<\infty,$
2. $W\left(F\left(x;\xi\right)\right)$ is differentiable and monotonically non-decreasing.
3. $W\left(F\left(x;\xi\right)\right)\to a, as\ x\to-\infty$, and $W\left(F\left(x;\xi\right)\right)\to b$,as $x\to\infty$, i.e $W\left(F\left(x;\xi\right)\right)\to 0$,as $x\to 0$, and $W\left(F\left(x;\xi\right)\right)\to 1$,as $x\to\infty$.

So the CDF is

$$G\left(x\right)_{OBXII-G}=\int_0^{W(F(x;\xi))}\delta\gamma t^{\gamma-1}\left[1+t^{\gamma}\right]^{-(\delta+1)}dt$$

$$G\left(x\right)_{OBXII-G}=1-\left[1+\left[\frac{\left[F(x;\xi)\right]^2}{1-F\left(x;\xi\right)}\right]^{\gamma}\right]^{-\delta} \tag{1}$$

The PDF of the OBXII –G Family is

$$g\left(x\right)_{OBXII-G}=\delta\gamma F(x;\xi)f(x;\xi)\left(2-F(x;\xi)\right)\left(1-F(x;\xi)\right)^{-2}\left[\frac{\left[F(x;\xi)\right]^2}{1-F(x;\xi)}\right]^{\gamma-1}\left[1+\left[\frac{\left[F(x;\xi)\right]^2}{1-F(x;\xi)}\right]^{\gamma}\right]^{-(\delta+1)} \tag{2}$$

2. The odd Burr XII Gompertz distribution (OBXIIGo)

Consider the Gompertz (Go) Distribution with shape parameters (a) and (b). Then CDF and PDF of the Go distribution are defined as follows:

$$F\left(x;a,c\right)_{Go}=1-e^{-\frac{a}{c}\left(e^{cx}-1\right)} \tag{3}$$

And

$$f\left(x;a,c\right)_{Go}=ae^{cx}e^{-\frac{a}{c}\left(e^{cx}-1\right)},x,a,b>0 \tag{4}$$

The Odd Burr XII Gompertz (OBXIIGo) Distribution is a novel distribution proposed by us. We substitute Eq.(3) into Eq.(1) to derive the OBXIIGo distribution. This results in the following expression.

$$G\left(x\right)_{OBXIIGo}=1-\left[1+\left[\frac{\left(1-e^{-\frac{a}{c}\left(e^{cx}-1\right)}\right)^2}{e^{-\frac{a}{c}\left(e^{cx}-1\right)}}\right]^{\gamma}\right]^{-\delta} \tag{5}$$

Corresponding PDF is

$$g\left(x\right)_{OBXIIGo} = \delta\gamma a e^{cx} e^{\frac{a}{c}\left(e^{cx}-1\right)} \left(1-e^{-\frac{2a}{c}\left(e^{cx}-1\right)}\right) \left[\frac{\left(1-e^{-\frac{a}{c}\left(e^{cx}-1\right)}\right)^2}{e^{-\frac{a}{c}\left(e^{cx}-1\right)}}\right]^{\gamma-1} \left[1+\left[\frac{\left(1-e^{-\frac{a}{c}\left(e^{cx}-1\right)}\right)^2}{e^{-\frac{a}{c}\left(e^{cx}-1\right)}}\right]^{\gamma}\right]^{-(\delta+1)} \tag{6}$$

Where $x, \delta, \gamma, a, b > 0$

$$h\left(x\right)_{OBXIIGo} = \frac{\delta\gamma a e^{cx} e^{\ \frac{a}{c}\left(e^{cx}-1\right)} \left(1-e^{-\frac{2a}{c}\left(e^{cx}-1\right)}\right) \left[\frac{\left(1-e^{-\frac{a}{c}\left(e^{cx}-1\right)}\right)^2}{e^{-\frac{a}{c}\left(e^{cx}-1\right)}}\right]^{\gamma-1}}{1+\left[\frac{\left(1-e^{-\frac{a}{c}\left(e^{cx}-1\right)}\right)^2}{e^{-\frac{a}{c}\left(e^{cx}-1\right)}}\right]^{\gamma}} \tag{7}$$

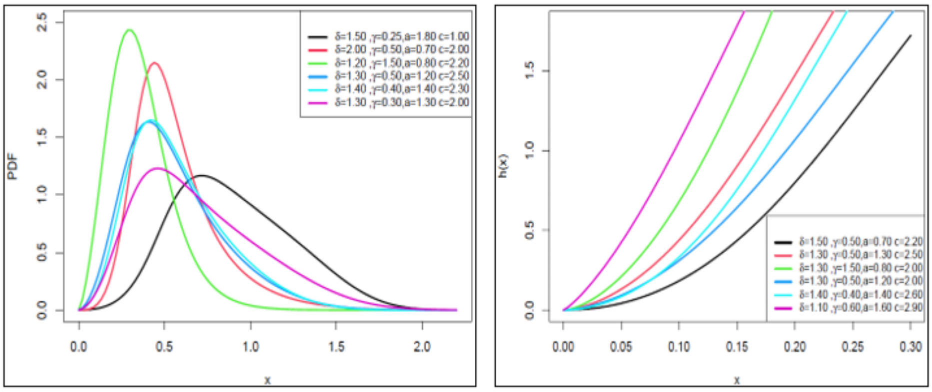

Figure 1: The plots of the PDF and h(x) of the OBXIIGo distribution

3. Statistical properties for OBXIIGo distribution

3.1. Useful representations of CDF and PDF

In this subsection, we aim to simplify Eq.(1) by utilizing two expansions: the Exponential expansion $e^{-p} = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} p^j$ and binomial series expansion $[1-u]^p = \sum_{h=0}^{\infty} (-1)^h \binom{p}{h} u^h, \quad [1-u]^{-p} = \sum_{m=0}^{\infty} \frac{\Gamma(p+j)}{m!\Gamma(p)} u^m$: $|u| < 1, \ p > 0$. Can be expanded to provide the following representation [13, 14]:

$$G\left(x\right) = 1 - ¥\left[F\left(x\right)\right]^j$$

Where

$$¥ = \sum_{r=i=m=j=0}^{\infty} \frac{(-1)^{r+m+j}\,\Gamma(\gamma r+i)}{i!\Gamma(\gamma r)} \binom{\delta+r-1}{r} \binom{2\gamma r+i}{m} \binom{m}{j}$$

By substituting the Eq.(3) into equation above, we get

$$G\left(x\right) = ¥\left[1-e^{-\frac{a}{c}\left(e^{cx}-1\right)}\right]^j$$

By using binomial series expansion:

$$\left[1-e^{-\frac{a}{c}\left(e^{cx}-1\right)}\right]^{j}=\sum_{w=0}^{\infty}(-1)^{w}\binom{j}{w}e^{-\frac{a}{c}\left(e^{cx}-1\right)w}$$

Then

$$G\left(x\right)=\yen\sum_{w=0}^{\infty}(-1)^{w}\binom{j}{w}e^{-\frac{a}{c}\left(e^{cx}-1\right)w}$$

And using exponential expansion:

$$e^{-\frac{a}{c}\left(e^{cx}-1\right)w}=\sum_{h=0}^{\infty}\frac{(-1)^{2h}\,a^{h}w^{h}}{h!c^{h}}\left(1-e^{cx}\right)^{h}$$

Then

$$G\left(x\right)=\yen\sum_{w=h=0}^{\infty}\frac{(-1)^{w+2h}\,a^{h}w^{h}}{h!c^{h}}\binom{j}{w}\left(1-e^{cx}\right)^{h}$$

Again, using binomial series expansion, we get

$$\left(1-e^{cx}\right)^{h}=\sum_{z=0}^{\infty}(-1)^{z}\binom{h}{z}e^{czx}$$

Then the CDF of OBXIIGo after representation:

$$G\left(x\right)_{OBXIIGo}=\eta e^{czx} \tag{8}$$

Where

$$\eta=\yen\sum_{w=h=z=0}^{\infty}\frac{(-1)^{w+2h+z}\,a^{h}w^{h}}{h!c^{h}}\binom{j}{w}\binom{h}{z}$$

Now we aim to simplify Eq.(2) by utilizing Exponential expansion and binomial series expansion, we get

$$g\left(x\right)=2€\left(x\right)\left[F\left(x\right)\right]^{m}-€f\left(x\right)\left[F\left(x\right)\right]^{m+1}$$

$$€=\sum_{z=w=i=m=0}^{\infty}\frac{(-1)^{z+i+m}\,\gamma\delta\Gamma\left(\gamma z+\gamma+1+w\right)}{w!\Gamma(\gamma z+\gamma+1)}\binom{\delta+r}{z}\binom{w+2\gamma z+2\gamma-1}{i}\binom{i}{m}$$

By substituting the Eqs.(3) and (4) into equation above, we get

$$g\left(x\right)=2€ae^{cx}e^{-\frac{a}{c}\left(e^{cx}-1\right)}\left[1-e^{-\frac{a}{c}\left(e^{cx}-1\right)}\right]^{j}-€ae^{cx}e^{-\frac{a}{c}\left(e^{cx}-1\right)}\left[1-e^{-\frac{a}{c}\left(e^{cx}-1\right)}\right]^{j+1}$$

By using binomial series expansion

$$\left[1-e^{-\frac{a}{c}\left(e^{cx}-1\right)}\right]^{j}=\sum_{r=0}^{\infty}(-1)^{r}\binom{j}{r}e^{-\frac{a}{c}\left(e^{cx}-1\right)r},\ \left[1-e^{-\frac{a}{c}\left(e^{cx}-1\right)}\right]^{j+1}=\sum_{v=0}^{\infty}(-1)^{v}\binom{j+1}{v}e^{-\frac{a}{c}\left(e^{cx}-1\right)v}$$

Then

$$g\left(x\right)=2€\sum_{r=0}^{\infty}(-1)^{r}\binom{j}{r}ae^{\frac{a}{c}(1+r)}e^{cx}e^{-\frac{a}{c}(1+r)e^{cx}}-€\sum_{v=0}^{\infty}(-1)^{v}\binom{j+1}{v}ae^{\frac{a}{c}(1+v)}e^{cx}e^{-\frac{a}{c}(1+v)e^{cx}}$$

And using exponential expansion

$$e^{-\frac{a}{c}(1+r)e^{cx}}=\sum_{m=0}^{\infty}\frac{(-1)^{m}\,a^{m}\left(1+r\right)^{m}}{m!c^{m}}e^{cmx},\ e^{-\frac{a}{c}(1+v)e^{cx}}=\sum_{s=0}^{\infty}\frac{(-1)^{s}\,a^{s}\left(1+v\right)^{s}}{s!c^{s}}e^{csx}$$

Then the PDF of OBXIIGo after representation:

$$g\left(x\right)_{OBXIIGo} = \mathrm{T}e^{c(m+1)x} - Pe^{c(s+1)x} \tag{9}$$

Where

$$\mathrm{T} = \frac{2A\sum_{r=m=0}^{\infty}\left(-1\right)^{r+m}\binom{j}{r}a^{1+m}e^{\;\frac{a}{c}(1+r)}\left(1+r\right)^{m}}{m!c^{m}},\; P = \frac{A\sum_{v=s=0}^{\infty}\left(-1\right)^{v+s}\binom{j+1}{v}a^{1+s}e^{\;\frac{a}{c}(1+v)}\left(1+v\right)^{s}}{s!c^{s}}$$

3.2. Quantile function

We can find the quantile function of the OBXIIGo distribution. It can be done by using the inverse function of the cumulative distribution function (CDF). The CDF is defined in Eq.(3). By using algebraic operations, we can solve for the random variable. This will give us the following formula.

$$Q(u) = \frac{1}{c}ln\left(1-\frac{c}{a}ln\left(1-\frac{-\left[\left[1-u\right]^{-\frac{1}{\delta}}-1\right]^{\frac{1}{\gamma}}+\sqrt{\left[\left[1-u\right]^{-\frac{1}{\delta}}-1\right]^{\frac{2}{\gamma}}+4\left[\left[1-u\right]^{-\frac{1}{\delta}}-1\right]^{\frac{1}{\gamma}}}}{2}\right)\right) \tag{10}$$

Table 1: Quantiles for Selected Parameter Values of the OBXIIGo Distribution

	(γ,δ,a,c)				
u	(0.7,2.7,1.3,1.7)	(3.1,2.3,0.9,2.4)	(1.5,2.3,1.6,1.5)	(0.8,2,1.8,0.6)	(2.1,3.4,1.2,0.8)
0.1	0.3823	0.3418	0.2807	0.3054	0.4416
0.2	0.4255	0.3813	0.3210	0.3658	0.4843
0.3	0.4564	0.4078	0.3493	0.4118	0.5134
0.4	0.4833	0.4291	0.3732	0.4533	0.5372
0.5	0.5093	0.4481	0.3954	0.4946	0.5589
0.6	0.5367	0.4664	0.4178	0.5393	0.5801
0.7	0.5681	0.4852	0.4421	0.5919	0.6027
0.8	0.6086	0.5066	0.4715	0.6615	0.6291
0.9	0.6732	0.5357	0.5150	0.7763	0.6667

4. Moment and associated measures

If a random variable X follows the OBXIIGo distribution, the ordinary moments of X can be computed as follows [15]:

$$\dot{\mu}_n = E(X^n) = \int_0^\infty x^n g\left(x,\delta,\gamma,a,c\right)dx$$

Applying (9) into the equation above we get:

$$\dot{\mu}_n = E\left(X^n\right) = \mathrm{T}\int_0^\infty x^n e^{c(m+1)x}dx - P\int_0^\infty x^n e^{c(s+1)x}dx$$

Let

$$-u = c\left(m+1\right)x \;\Rightarrow x = \frac{-u}{c\left(m+1\right)},$$

then

$$\frac{-du}{dx} = c\left(m+1\right) \Rightarrow dx = -\frac{du}{\gimel\left(k+1\right)}$$

Then

$$\acute{\mu}_n = E\left(X^n\right) = \mathrm{T}\left(-\frac{1}{c\left(m+1\right)}\right)^{n+1}\int_0^{\infty} u^n e^{-u} du - P\left(-\frac{1}{c(s+1)}\right)^{n+1}\int_0^{\infty} u^v e^{-u} dv$$

After using gamma integral, we have derived the following final expression:

$$\acute{\mu}_n = E\left(X^n\right) = \mathrm{T}\Gamma(n+1)\left(-\frac{1}{c\left(m+1\right)}\right)^{n+1} - P\Gamma(n+1)\left(-\frac{1}{c(s+1)}\right)^{n+1} \tag{11}$$

By utilizing Eq.(11), we can compute the first four moments about the origin. The resulting expressions for these moments are as follows:

$$\acute{\mu}_1 = E\left(X^1\right) = \mathrm{T}\left(-\frac{1}{c\left(m+1\right)}\right)^{2} - P\left(-\frac{1}{c(s+1)}\right)^{2} \tag{12}$$

$$\acute{\mu}_2 = E\left(X^2\right) = 2\mathrm{T}\left(-\frac{1}{c\left(m+1\right)}\right)^{3} - 2P\left(-\frac{1}{c(s+1)}\right)^{3} \tag{13}$$

$$\acute{\mu}_3 = E\left(X^3\right) = 6\mathrm{T}\left(-\frac{1}{c\left(m+1\right)}\right)^{4} - 6P\left(-\frac{1}{c(s+1)}\right)^{4} \tag{14}$$

And

$$\acute{\mu}_4 = E\left(X^4\right) = 24\mathrm{T}\left(-\frac{1}{c\left(m+1\right)}\right)^{5} - 24P\left(-\frac{1}{c\left(s+1\right)}\right)^{5} \tag{15}$$

We can calculate the variance, skewness, and kurtosis of the OBXIIGo distribution using the derived moments. These measures tell us about the distribution's characteristics. The values for the variance, skewness, and kurtosis of the OBXIIGo distribution are:

$$Var(X) = \left(2\mathrm{T}\left(-\frac{1}{c\left(m+1\right)}\right)^{3} - 2P\left(-\frac{1}{c\left(s+1\right)}\right)^{3}\right) - \left(\mathrm{T}\left(-\frac{1}{c\left(m+1\right)}\right)^{2} - P\left(-\frac{1}{c(s+1)}\right)^{2}\right)^{2} \tag{16}$$

$$SK = \frac{\mu_3}{\mu_2^{\left(\frac{3}{2}\right)}} = \frac{6\mathrm{T}\left(-\frac{1}{c(m+1)}\right)^{4} - 6P\left(-\frac{1}{c(s+1)}\right)^{4}}{\left(2\mathrm{T}\left(-\frac{1}{c(m+1)}\right)^{3} - 2P\left(-\frac{1}{c(s+1)}\right)^{3}\right)^{\frac{3}{2}}} \tag{17}$$

$$KU = \frac{\mu_4}{\mu_2^2} = \frac{24\mathrm{T}\left(-\frac{1}{c(m+1)}\right)^{5} - 24P\left(-\frac{1}{c(s+1)}\right)^{5}}{\left(2\mathrm{T}\left(-\frac{1}{c(m+1)}\right)^{3} - 2P\left(-\frac{1}{c(s+1)}\right)^{3}\right)^{2}} \tag{18}$$

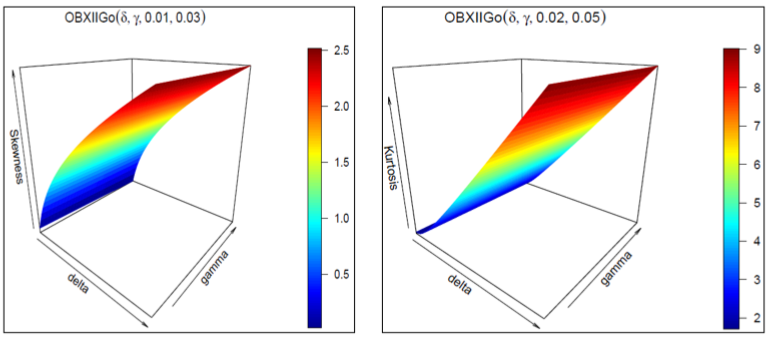

Figure 2: Displays 3D plots illustrating the skewness and kurtosis of the OBXIIGo distribution.

4.1. Order statistics

Consider a random sample $X_1, X_2, \ldots, X_n$ of size n from the OBXIIGo distribution. Let $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ represent the order statistics of this sample. The probability density function (PDF) for the order statistics can be described as follows:

$$g_{q:n}(x) = \Omega \sum_{s=0}^{n-q} (-1)^s \binom{n-q}{s} \left[G(x)_{OBXIIGo}\right]^{q+s-1} g(x)_{OBXIIGo}$$

Where

$$\Omega = \frac{n!}{(q-1)!\,(q-p)!}$$

Substituting Eqs.(5) and (6) into the above equation, we get:

$$g_{q:n}(x) = \Omega \sum_{s=0}^{n-q} (-1)^s \binom{n-q}{s} \left[1 - \left[1 + \left[\frac{\left(1 - e^{-\frac{a}{c}(e^{cx}-1)}\right)^2}{e^{-\frac{a}{c}(e^{cx}-1)}}\right]^{\gamma}\right]^{-\delta}\right]^{q+s-1}$$
$$\left[\delta\gamma a e^{cx} e^{\frac{a}{c}(e^{cx}-1)} \left(1 - e^{-\frac{2a}{c}(e^{cx}-1)}\right) \left[\frac{\left(1 - e^{-\frac{a}{c}(e^{cx}-1)}\right)^2}{e^{-\frac{a}{c}(e^{cx}-1)}}\right]^{\gamma-1} \left[1 + \left[\frac{\left(1 - e^{-\frac{a}{c}(e^{cx}-1)}\right)^2}{e^{-\frac{a}{c}(e^{cx}-1)}}\right]^{\gamma}\right]^{-(\delta+1)}\right]$$

The PDF of the minimum order statistic, denoted as $g_{1:n}(x)$, when $q = 1$, and the PDF of the maximum order statistic, denoted as $g_{n:n}(x)$, when $q = n$, for the OBXIIGo distribution can be expressed as follows:

$$g_{1:n}(x) = n \sum_{s=0}^{n-1} (-1)^s \binom{n-1}{s} \left[1 - \left[1 + \left[\frac{\left(1 - e^{-\frac{a}{c}(e^{cx}-1)}\right)^2}{e^{-\frac{a}{c}(e^{cx}-1)}}\right]^{\gamma}\right]^{-\delta}\right]^{s}$$
$$\left[\delta\gamma a e^{cx} e^{\frac{a}{c}(e^{cx}-1)} \left(1 - e^{-\frac{2a}{c}(e^{cx}-1)}\right) \left[\frac{\left(1 - e^{-\frac{a}{c}(e^{cx}-1)}\right)^2}{e^{-\frac{a}{c}(e^{cx}-1)}}\right]^{\gamma-1} \left[1 + \left[\frac{\left(1 - e^{-\frac{a}{c}(e^{cx}-1)}\right)^2}{e^{-\frac{a}{c}(e^{cx}-1)}}\right]^{\gamma}\right]^{-(\delta+1)}\right]$$

$$g_{n:n}(x) = n \left[1 - \left[1 + \left[\frac{\left(1 - e^{-\frac{a}{c}(e^{cx}-1)}\right)^2}{e^{-\frac{a}{c}(e^{cx}-1)}}\right]^{\gamma}\right]^{-\delta}\right]^{n+s-1}$$
$$\left[\delta\gamma a e^{cx} e^{\frac{a}{c}(e^{cx}-1)} \left(1 - e^{-\frac{2a}{c}(e^{cx}-1)}\right) \left[\frac{\left(1 - e^{-\frac{a}{c}(e^{cx}-1)}\right)^2}{e^{-\frac{a}{c}(e^{cx}-1)}}\right]^{\gamma-1} \left[1 + \left[\frac{\left(1 - e^{-\frac{a}{c}(e^{cx}-1)}\right)^2}{e^{-\frac{a}{c}(e^{cx}-1)}}\right]^{\gamma}\right]^{-(\delta+1)}\right]$$

4.2. Rényi entropy

The Rényi entropy of the OBXIIGo distribution is defined in the following manner:

$$T_R(\varphi)_{OBXIIGo} = \frac{1}{1-\varphi} log \int_0^{\infty} g^{\varphi}(x)_{OBXIIGo}\, dx, \quad \varphi > 0, \ \varphi \neq 1$$

By substituting the Eq.(9) into the above equation, we get

$$T_R(\varphi)_{OBXIIGo} = \frac{1}{1-\varphi} \log\left(\int_0^{\infty} \left(Te^{c(m+1)x} - Pe^{c(s+1)x}\right)^{\varphi} dx\right) \tag{19}$$

5. Maximum likelihood estimation

Consider a random sample of size n, denoted as $X_1, X_2, \ldots, X_n$, drawn from the OBXIIGo distribution with parameters γ, δ, a, and c. Let $€ = (\gamma, \delta, a, c)^{\mathrm{T}}$ represent the vector of these parameters. The log-likelihood (LL) function for this distribution can be expressed as follows:

$$\begin{aligned} l = {} & n\log\delta + n\log\gamma + n\log a + \sum_{i=1}^{n} cx_i + \frac{a}{c}\sum_{i=1}^{n}\left(e^{cx_i}-1\right) + \sum_{i=1}^{n} log\left(1-e^{-\frac{2a}{c}\left(e^{cx_i}-1\right)}\right) \\ & + (\gamma-1)\sum_{i=1}^{n}\log\left(\frac{\left(1-e^{-\frac{a}{c}\left(e^{cx_i}-1\right)}\right)^2}{e^{-\frac{a}{c}\left(e^{cx_i}-1\right)}}\right) \\ & - (\delta+1)\sum_{i=1}^{n}\log\left(1+\left[\frac{\left(1-e^{-\frac{a}{c}\left(e^{cx_i}-1\right)}\right)^2}{e^{-\frac{a}{c}\left(e^{cx_i}-1\right)}}\right]^{\gamma}\right) \end{aligned} \tag{20}$$

The first partial derivatives of the log-likelihood function with respect to the four parameters, namely γ, δ, a, and c, can be expressed as follows:

$$\frac{\partial(l)}{\partial\delta} = \frac{n}{\delta} - \sum_{i=1}^{n}\log\left(1+\left[\frac{\left(1-e^{-\frac{a}{c}\left(e^{cx_i}-1\right)}\right)^2}{e^{-\frac{a}{c}\left(e^{cx_i}-1\right)}}\right]^{\gamma}\right) \tag{21}$$

$$\frac{\partial(l)}{\partial\gamma} = \frac{n}{\gamma} + \sum_{i=1}^{n} cx_i\log\left(\frac{\left(1-e^{-\frac{a}{c}\left(e^{cx_i}-1\right)}\right)^2}{e^{-\frac{a}{c}\left(e^{cx_i}-1\right)}}\right) - \sum_{i=1}^{n}\frac{(\delta+1)\left[\frac{\left(1-e^{-\frac{a}{c}\left(e^{cx_i}-1\right)}\right)^2}{e^{-\frac{a}{c}\left(e^{cx_i}-1\right)}}\right]^{\gamma}\ln\left(\frac{\left(1-e^{-\frac{a}{c}\left(e^{cx_i}-1\right)}\right)^2}{e^{-\frac{a}{c}\left(e^{cx_i}-1\right)}}\right)}{1+\left[\frac{\left(1-e^{-\frac{a}{c}\left(e^{cx_i}-1\right)}\right)^2}{e^{-\frac{a}{c}\left(e^{cx_i}-1\right)}}\right]^{\gamma}} \tag{22}$$

$$\begin{aligned} \frac{\partial\,(l)}{\partial a} = {} & \frac{n}{a} + \frac{1}{c}\sum_{i=1}^{n}\left(e^{cx_i}-1\right) + \sum_{i=1}^{n}\frac{2\left(e^{cx_i}-1\right)e^{-\frac{2a}{c}\left(e^{cx_i}-1\right)}}{c\left(1-e^{-\frac{2a}{c}\left(e^{cx_i}-1\right)}\right)} \\ & + (\gamma-1)\sum_{i=1}^{n}\frac{\left(\frac{2}{c}\left(e^{cx_i}-1\right)\left(1-e^{-\frac{2a}{c}\left(e^{cx_i}-1\right)}\right) + \frac{\left(e^{cx_i}-1\right)\left(1-e^{-\frac{a}{c}\left(e^{cx_i}-1\right)}\right)^2}{ce^{-\frac{a}{c}\left(e^{cx_i}-1\right)}}\right)e^{-\frac{a}{c}\left(e^{cx_i}-1\right)}}{\left(1-e^{-\frac{a}{c}\left(e^{cx_i}-1\right)}\right)^2} \\ & - (\delta+1)\sum_{i=1}^{n}\frac{\left[\frac{\left(1-e^{-\frac{a}{c}\left(e^{cx_i}-1\right)}\right)^2}{e^{-\frac{a}{c}\left(e^{cx_i}-1\right)}}\right]^{\gamma}\left(\frac{2}{c}\left(e^{cx_i}-1\right)\left(1-e^{-\frac{2a}{c}\left(e^{cx_i}-1\right)}\right) + \frac{\left(e^{cx_i}-1\right)\left(1-e^{-\frac{a}{c}\left(e^{cx_i}-1\right)}\right)^2}{ce^{-\frac{a}{c}\left(e^{cx_i}-1\right)}}\right)\gamma e^{-\frac{a}{c}\left(e^{cx_i}-1\right)}}{\left(1-e^{-\frac{a}{c}\left(e^{cx_i}-1\right)}\right)^2\left(1+\left[\frac{\left(1-e^{-\frac{a}{c}\left(e^{cx_i}-1\right)}\right)^2}{e^{-\frac{a}{c}\left(e^{cx_i}-1\right)}}\right]^{\gamma}\right)} \end{aligned} \tag{23}$$

$$
\begin{aligned}
\frac{\partial (l)}{\partial c} &= \sum_{i=1}^{n} x_i - \frac{a}{c^2}\sum_{i=1}^{n} e^{cx_i} - 1\left(cx_i - 1\right) + \frac{2a}{c}\sum_{i=1}^{n}\frac{\left(x_i e^{cx_i} - \frac{\left(e^{cx_i}-1\right)}{c}\right) e^{-\frac{2a}{c}\left(e^{cx_i}-1\right)}}{\left(1 - e^{-\frac{2a}{c}\left(e^{cx_i}-1\right)}\right)} \\
&+ \sum_{i=1}^{n}\frac{\left(\gamma - 1\right) e^{-\frac{a}{c}\left(e^{cx_i}-1\right)}}{\left(1 - e^{-\frac{a}{c}\left(e^{cx_i}-1\right)}\right)^2} \\
&\left(\frac{2a}{c}\left(\left(1 - e^{-\frac{a}{c}\left(e^{cx_i}-1\right)}\right)\left(x_i - \frac{cx_i - 1}{c}\right)\right) - \frac{\left(1 - e^{-\frac{a}{c}\left(e^{cx_i}-1\right)}\right)^2\left(-\frac{x_i}{c} + \frac{\left(cx_i-1\right)a}{c^2}\right)}{e^{-\frac{a}{c}\left(e^{cx_i}-1\right)}}\right) \\
&- \left(\delta + 1\right)\sum_{i=1}^{n}\frac{\gamma e^{-\frac{a}{c}\left(e^{cx_i}-1\right)}\left[\frac{\left(1-e^{-\frac{a}{c}\left(e^{cx_i}-1\right)}\right)^2}{e^{-\frac{a}{c}\left(e^{cx_i}-1\right)}}\right]^{\gamma}}{\left(1 - e^{-\frac{a}{c}\left(e^{cx_i}-1\right)}\right)^2\left(1 + \left[\frac{\left(1-e^{-\frac{a}{c}\left(e^{cx_i}-1\right)}\right)^2}{e^{-\frac{a}{c}\left(e^{cx_i}-1\right)}}\right]^{\gamma}\right)} \\
&\frac{\left(\left(\frac{2a}{c}\left(\left(1 - e^{-\frac{a}{c}\left(e^{cx_i}-1\right)}\right)\left(x_i - \frac{cx_i-1}{c}\right)\right)\right) - \frac{\left(1-e^{-\frac{a}{c}\left(e^{cx_i}-1\right)}\right)^2\left(-\frac{x_i}{c}+\frac{\left(cx_i-1\right)a}{c^2}\right)}{e^{-\frac{a}{c}\left(e^{cx_i}-1\right)}}\right)}{\left(1 - e^{-\frac{a}{c}\left(e^{cx_i}-1\right)}\right)^2\left(1 + \left[\frac{\left(1-e^{-\frac{a}{c}\left(e^{cx_i}-1\right)}\right)^2}{e^{-\frac{a}{c}\left(e^{cx_i}-1\right)}}\right]^{\gamma}\right)}
\end{aligned} \tag{24}
$$

Calculating Eqs.21, 22, 23, and 24 manually to find the values of the parameters when they equal zero can be challenging. In this study, it is recommended to use statistical software such as R to solve these equations numerically.

6. Simulation study

In this section, we present the outcomes of a simulation study conducted for the OBXIIGo distribution. The study explores eight sets of parameter values: (δ=1.1, γ=1.1, a=0.7, c=0.7), (δ=1.1, γ=1.1, a=2.1, c=0.7), (δ=1.2, γ=1.2, a=1.4, c=1.4), (δ=0.6, γ=1.2, a=0.5, c=1.4). For each parameter set, we generate a total of N = 1000 samples and consider sample sizes of n = 75, 150, 200, and 400. Based on these samples, an estimate of the mean, average bias, and root mean square error (RMSE) is found. The bias and RMSE of the estimated parameter, denoted by $\hat{\delta}$, are also calculated according to the following relationships:

$$
Abias\left(\hat{\delta}\right) = \frac{\sum_{i=1}^{N}\hat{\delta}_i}{N} - \delta, \text{ and } RMSE\left(\hat{\delta}\right) = \sqrt{\frac{\sum_{i=1}^{N}\left(\hat{\delta}_i - \delta\right)^2}{N}}.
$$

According to Tables 2 and 3, it appears that the average values of the estimated parameters appear to be very close to the real parameter values. Moreover, the RMSE and bias values continually decrease toward zero for all simulated parameter values. It also appears that the OBXIIGo model produces consistent MLEs for the parameters.

Table 2: Monte Carlo simulation results for the OBXIIGo distribution

		($\delta = 1.1, \gamma = 1.1, a = 0.7, c = 0.7$)			($\delta = 1.1, \gamma = 0.4, a = 2.1, c = 0.7$)		
parameter	Sample Size	Mean	RMSE	Abias	Mean	RMSE	Abias
δ	75	1.5776	4.1948	0.4776	1.4628	5.6229	0.3628
	150	1.5453	3.9803	0.4453	1.1365	2.8075	0.0365
	200	1.5574	3.5394	0.4374	1.1039	1.6300	0.0039
	400	1.2499	1.5130	0.1499	1.0709	0.6423	-0.0290
γ	75	1.2212	0.6328	0.1212	1.2136	0.6337	0.1136
	150	1.1877	0.2623	0.0877	1.1828	0.2516	0.0828
	200	1.1567	0.2279	0.0567	1.1583	0.2156	0.0583
	400	1.1254	0.1065	0.0254	1.1254	0.1041	0.0254
a	75	0.8905	0.5289	0.1905	2.7007	1.5192	0.6007
	150	0.8547	0.3893	0.1547	2.6269	1.1551	0.5269
	200	0.8028	0.3004	0.1028	2.4863	0.9080	0.3863
	400	0.7618	0.2124	0.0618	2.3234	0.6387	0.2234
c	75	0.9153	0.4616	0.2153	1.2453	0.8790	0.5453
	150	0.8429	0.3923	0.1429	1.0622	0.6876	0.3622
	200	0.8063	0.3587	0.1063	0.9918	0.6047	0.2918
	400	0.7576	0.2660	0.0576	0.8517	0.4198	0.1517

Table 3: Monte Carlo simulation results for the OBXIIGo distribution

		($\delta = 1.2, \gamma = 1.2, a = 1.4, c = 1.4$)			($\delta = 0.6, \gamma = 1.2, a = 0.5, c = 1.4$)		
parameter	Sample Size	Mean	RMSE	Abias	Mean	RMSE	Abias
δ	75	3.2420	13.543	2.0420	0.9288	2.0000	0.3288
	150	2.0669	8.0777	0.8669	0.8009	1.8118	0.2509
	200	1.7691	5.7882	0.5691	0.8458	1.9458	0.2458
	400	1.2438	1.1855	0.0438	0.7291	0.9350	0.1291
γ	75	1.2761	0.3948	0.0761	1.3703	0.4927	0.1703
	150	1.2631	0.2357	0.0631	1.3208	0.3374	0.1208
	200	1.2522	0.2202	0.0522	1.2824	0.2244	0.0824
	400	1.2171	0.1132	0.0171	1.2390	0.1394	0.0390
a	75	1.6430	0.8486	0.2430	0.5682	0.2467	0.0682
	150	1.5940	0.5653	0.1940	0.5533	0.1739	0.0533
	200	1.5670	0.5427	0.1670	0.5399	0.1509	0.0399
	400	1.4995	0.3297	0.0995	0.5202	0.1021	0.0202
c	75	1.9652	1.0766	0.5652	1.4477	0.4837	0.0477
	150	1.7994	0.9240	0.3994	1.4214	0.3963	0.0214
	200	1.7378	0.8449	0.3378	1.4087	0.3826	0.0087
	400	1.6124	0.6574	0.2124	1.3938	0.2861	-0.0061

7. Application

In this section, it is done Perform an appropriate analysis of the OBXIIGo distribution for two data sets. Then it is compared with other distributions, which are shown in Table 4.

To evaluate the OBXIIGo model and comparative models, we use eight widely known measures of goodness of fit. These measures include the Akaike information criterion (AIC), consistent AIC (CAIC), the Bayesian information criterion (BIC), the Hannan-Quinn information criterion (HQIC), the Anderson-Darling statistic (A), the Kolmogorov-Smirnov statistic (KS), the Cramer-von Mises statistic (W), and the corresponding p-value for the KS test.

In Tables 5 and 6, it is evident that the OLE distribution demonstrates the smallest values of AIC, AICC, and BIC in comparison to the corresponding values for the non-nested distributions. Additionally, the goodness-of-fit statistics A, W, Kolmogorov-Smirnov (KS) test, and its associated p-value all indicate that the OLE distribution provides the best fit for both Data I and Data II.

Table 4: Comparative distributions

Distribution	CDF
Truncated exponentiated exponential gompertz distribution (TEEGo) [16]	$\left(1-e^{-\gamma\left(1-e^{-\frac{a}{c}\left(e^{cx}-1\right)}\right)}\right)^{\delta} / \left(1-e^{-\gamma}\right)^{\delta}$
Beta Gompertz distribution (BeGo) [17]	$p\gamma(1-e^{-\frac{a}{c}\left(e^{cx}-1\right)}, \gamma, \delta)$
Kumaraswamy Gompertz distribution (KuGo) [18]	$1-\left(1-\left(1-e^{-\frac{a}{c}\left(e^{cx}-1\right)}\right)^{\gamma}\right)^{\delta}$
Exponential Generalized Gompertz distribution (EGGo) (New)	$\left(1-\left(1-\left(1-e^{-\frac{a}{c}\left(e^{cx}-1\right)}\right)\right)^{\gamma}\right)^{\delta}$
Weibull Gompertz distribution (WeGo) (New)	$(1-\exp\left(-\gamma^{-\delta}\ \left(e^{-\frac{a}{c}\left(e^{cx}-1\right)}\right)^{\delta}\right)$
Gompertz Gompertz distribution (GoGo) (New)	$1-e^{\frac{\gamma}{\delta}\left(1-\left(1-\left(1-e^{-\frac{a}{c}\left(e^{cx}-1\right)}\right)\right)^{-\delta}\right)}$
Rayleigh Gompertz distribution (RE) (New)	$e^{\frac{-\gamma}{2}\left(-\ln\left(1-e^{-\frac{a}{c}\left(e^{cx}-1\right)}\right)\right)^{2}}$
Burr type X (Bx) [19]	$\left(1-exp\left(-\left(cx\right)^{2}\right)\right)^{a}$

7.1. The First Dataset I

The dataset comprises of 74 observations, specifically referring to gauge lengths of 20 mm. The analysis of this data was performed by [20]:
(1.312, 1.314, 1.479 ,1.552,1.700 ,1.803, 1.861 ,1.865 ,1.944, 1.958 ,1.966, 1.997 ,2.006, 2.021 ,2.027, 2.055, 2.063 ,2.098, 2.140, 2.179 ,2.224 ,2.240, 2.253 ,2.270, 2.272, 2.274, 2.301, 2.301 ,2.359 ,2.382 ,2.382 ,2.426 ,2.434, 2.435, 2.478 ,2.490, 2.511, 2.514, 2.535 ,2.554, 2.566, 2.570, 2.586, 2.629 ,2.633, 2.642, 2.648, 2.684 ,2.697, 2.726, 2.770, 2.773, 2.800, 2.809, 2.818, 2.821, 2.848 ,2.880, 2.954, 3.012, 3.067 ,3.084, 3.090, 3.096, 3.128, 3.233, 3.433 ,3.585, 3.585).

Table 5: Estimates of models for data I

Dist.	MLEs	-2L	AIC	CAIC	BIC	HQIC	W	A	K-S	p-value
OBXIIGo	$\hat{\delta}$:4.1716 $\hat{\gamma}$:6.5503 $\hat{a}$:0.2690 $\hat{c}$:0.4175	48.88	105.76	106.38	114.69	109.30	0.0142	0.1297	0.0366	0.9999
TEEGo	$\hat{\delta}$:0.7878 $\hat{\gamma}$:1.6659 $\hat{a}$:1.4679 $\hat{c}$:0.0348	52.55	113.18	113.80	122.12	116.72	0.0383	0.3059	0.1109	0.3632
BeGo	$\hat{\delta}$:1.8252 $\hat{\gamma}$:0.3904 $\hat{a}$:1.6598 $\hat{c}$:0.0662	50.52	109.05	109.68	117.99	112.60	0.0501	0.3870	0.0625	0.9497

Table 5: Estimates of models for data I (*Continued*)

Dist.	MLEs	-2L	AIC	CAIC	BIC	HQIC	W	A	K-S	p-value
KuGo	$\hat{\delta}$:1.7485 $\hat{\gamma}$:0.4008 $\hat{a}$:1.7283 $\hat{c}$:0.0522	50.88	109.76	110.39	118.70	113.31	0.0581	0.4398	0.0698	0.8889
EGGo	$\hat{\delta}$:0.9552 $\hat{\gamma}$:1.7178 $\hat{a}$:1.4666 $\hat{c}$:0.0457	51.27	110.60	11.23	119.54	114.15	0.0529	0.4012	0.0781	0.7936
WeGo	$\hat{\delta}$:1.7897 $\hat{\gamma}$:1.0561 $\hat{a}$:1.0440 $\hat{c}$:0.0718	52.67	113.35	113.97	122.28	116.89	0.0922	0.6596	0.0829	0.7291
GoGo	$\hat{\delta}$:0.1953 $\hat{\gamma}$:0.5380 $\hat{a}$:0.8593 $\hat{c}$:0.2255	61.17	130.34	130.97	139.28	133.89	0.2549	1.6872	0.1505	0.0877
RGo	$\hat{\delta}$:1.2743 $\hat{\gamma}$:0.9415 $\hat{a}$:0.047	55.96	118.00	118.37	124.70	120.66	0.0199	0.1718	0.1399	0.1340
Bx	$\hat{a}$:0.6505 $\hat{c}$:7.7171	50.77	105.84	106.72	115.01	109.32	0.0548	0.3965	0.0723	0.8631

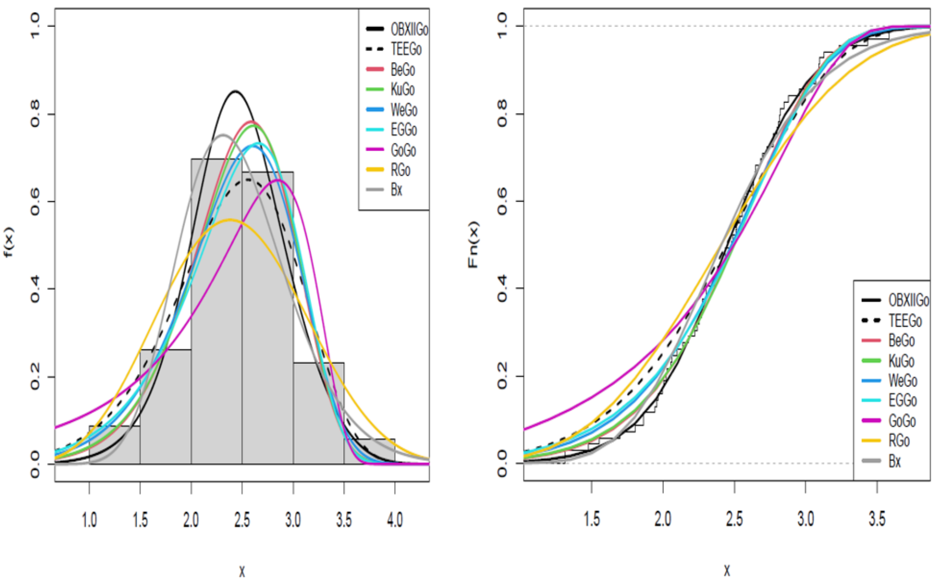

Figure 3: Estimated PDF, and CDF for Data I

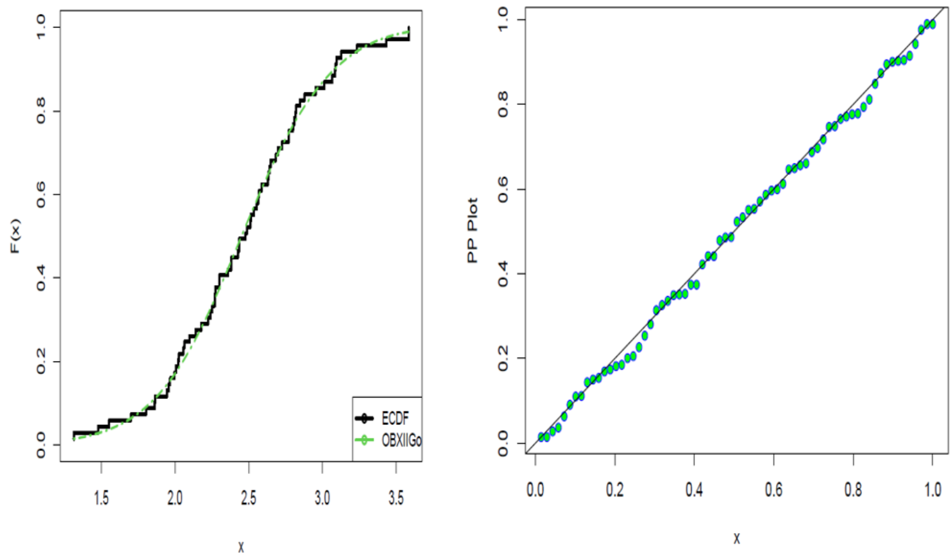

Figure 4: Estimated CDF, and P-P plot of OBXIIGo for Data I

7.2. The second dataset II

The study utilized a dataset consisting of the survival times of 72 guinea pigs that were infected with virulent tubercle bacilli. The survival times in this dataset were measured in days. The original observation and reporting of these data was performed by [21]:
(0.1, 0.33, 0.44, 0.56, 0.59, 0.59, 0.72, 0.74, 0.92, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 1.07, 1.08, 1.08, 1.08, 1.09, 1.12, 1.13, 1.15, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 1.36, 1.39, 1.44, 1.46, 1.53, 1.59, 1.6, 1.63, 1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16, 2.22, 2.3, 2.31, 2.4, 2.45, 2.51, 2.53, 2.54, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32, 4.58 ,5.55).

Table 6: Estimates of OBXIIGo and Comparative distribution for data I.

Dist.	MLEs	-2L	AIC	CAIC	BIC	HQIC	W	A	K-S	p-value
OBXIIGo	$\hat{\delta}$:1.3186 $\hat{\gamma}$:0.6616 $\hat{a}$:0.1542 $\hat{c}$:0.6832	93.38	194.77	195.36	203.87	198.39	0.0641	0.4016	0.0782	0.7704
TEEGo	$\hat{\delta}$:0.6825 $\hat{\gamma}$:3.2410 $\hat{a}$:0.1285 $\hat{c}$:0.7861	94.73	197.48	198.08	206.59	201.11	0.0998	0.6115	0.1005	0.4609
BeGo	$\hat{\delta}$:3.0302 $\hat{\gamma}$:0.9809 $\hat{a}$:0.0746 $\hat{c}$:0.9660	94.46	196.93	197.53	206.04	200.56	0.0932	0.5686	0.0981	0.4913
KuGo	$\hat{\delta}$:3.1652 $\hat{\gamma}$:1.5264 $\hat{a}$:0.0086 $\hat{c}$:0.8370	94.32	196.64	197.24	205.75	200.26	0.0837	0.5165	0.0925	0.5684

Table 6: Estimates of OBXIIGo and Comparative distribution for data I (*Continued*)

Dist.	MLEs	-2L	AIC	CAIC	BIC	HQIC	W	A	K-S	p-value
EGGo	$\hat{\delta}$:1.2277 $\hat{\gamma}$:3.0336 $\hat{a}$:0.0619 $\hat{c}$:0.7840	94.43	196.94	197.54	206.05	200.57	0.0910	0.5576	0.0990	0.4796
WeGo	$\hat{\delta}$:2.5627 $\hat{\gamma}$:0.9533 $\hat{a}$:0.3060 $\hat{c}$:0.6629	94.19	196.39	196.98	205.49	200.01	0.0821	0.4927	0.0894	0.6124
GoGo	$\hat{\delta}$:0.5397 $\hat{\gamma}$:0.0940 $\hat{a}$:0.7070 $\hat{c}$:0.4522	100.9	209.83	210.43	218.94	213.46	0.2455	1.4408	0.1538	0.0662
RGo	$\hat{\delta}$:1.1411 $\hat{\gamma}$:0.2639 $\hat{a}$:0.2077	95.54	197.09	197.44	203.92	199.81	0.1110	0.6955	0.1055	0.3991
Bx	$\hat{a}$:0.4767 $\hat{c}$:0.9263	96.45	196.91	197.08	204.46	198.72	0.1778	1.0393	0.0950	0.5333

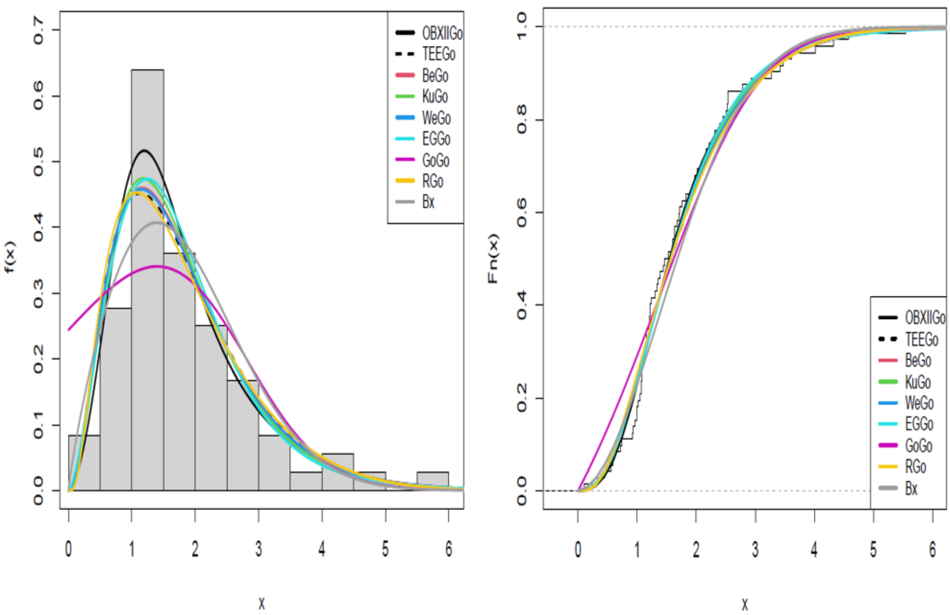

Figure 5: Estimated PDF, and CDF for Data II

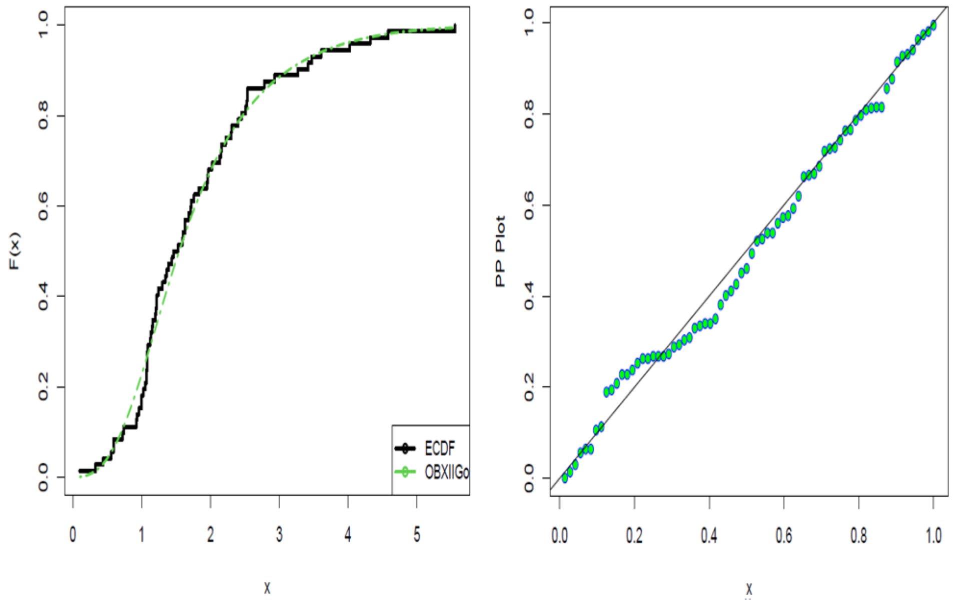

Figure 6: CDF and PP plot estimated for OBXIIGo for data II

8. Conclusion

This paper proposed to introduce a very flexible distribution called Odd Burr XII Gompertz distribution (OBXIIGo). A number of mathematical properties of the OBXIIGo distribution were analyzed. We used the maximum likelihood estimation technique to estimate the parameters of our new distribution, to ensure optimal estimation of the parameters. The results of the simulation study show that the maximum likelihood estimators of the OBXIIGo distribution are not only very accurate but also consistent. We applied it to two real data sets. A comparative analysis was performed to evaluate the performance of our new distribution OBXIIGo with the following statistical distributions: TEEGo, BeGo, KuGo, EGGo, WeGo, GoGo, RGo, and Bx. Results from analysis of real datasets indicate that the OBXIIGo distribution outperforms the compared distributions in terms of accuracy and fit.

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