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Using.Liouville's function for Creating a weird numbers from Reals

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Abstract

During 1937 Beurling Showed that any positive infinitely increasing real sequence such that the its first element precisely greater than one, called a Beurling's primes. Furthermore, the series of Beurling integers (or generalized integers) can be constructed using the fundamental theorem of arithmetic. During the seventieth of the last century, Diamond showed that majority of the arithmetical functions were generalized to deal with the generalization of the primes and integers. This work aims to create some weird numbers from a large enough reals So, the reader has to be familiar with Mobius inversion formula of the Pci function. The challenging of this work is the dealing with an algorithm for generating a weird numbers (or maybe a primitive weird numbers) from a large enough real numbers x. The idea of this work can be used for an application of modeling, data simulation and security subjects.

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2010 MSC: 49J20, 49K20, 35Q35, 93C20.

1. Introduction

Number theory is an important core of the pure mathematics subjects, where it deals either algebraically with the behaviors of the sequence of numbers and related counting functions or it deals analytically. For P takes an actual prime $\{2, 3, 5, \ldots\}$ one easly can build the sequence of related positive integers $N =$

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$\{1, 2, 3, \ldots\}$ by using the sense of the fundamental theorem of arithmetic on P.
This tells us that every number can be represented as a product of primes with a power (i.e. $p_1^{\beta_1} \cdot p_2^{\beta_2} \ldots p_k^{\beta_k}$ with β_k belongs to $N \cup \{0\}$ and k belongs to N). Many other authors have studied the application of the number theory in different topics. There is no loss of the generality if we write $N_0, \pi_0, \zeta_0 \ldots$ these are the Beurling Zeta function and the Beurling's counting function to let the reader know (of integers and primes).

2. Beurling's Prims:

For any real infinite sequence $r_1, r_2, \ldots$ for which $r_{1>1}, r_k \leq r_{k+1}$. So, Using the sence of the fundamental theorem of arithmetic, the sequence $\{1, n_1, n_2, n_3, \ldots\ldots\}$ can be build and it called the set of Beurling's integers. Define be the Beurling's g function of the set $N_0 \cdot N_g(x) = \sum_{n \leq x} 1$

3. Beurling's Prim Number Theorem:

Suppose for d > 0, that

$$N_0(x) = ax + \left(x(x)^{-1}\right) \tag{1}$$

Beurling (see [2]) proved that if $q > 3/2$, then

$$\pi_0(x) \sim \frac{x}{\log \log x} \tag{2}$$

Where

$$\pi_0(x) = \sum_{r_{jx}} 1.$$

In general the above called the generalization of the prime number theorem (PNT).

During demonstrated in the study (see [5]) in1970 that Beurling's condition is sufficiently acute that (1) exist for $b = 3/2$ and (2) fails to function. Diamond in (5) addresses the following issues in his compositions:

If (1) exists with b belongs to $(1/3, 2)$. Then (2) exists. There exist thus positive values r and R so that each and every $x > x_0$.

He did not give any examples for which (1) holds true with $b > 1$ but (2) fails. Later, Hall (see [6]) proved for each (r_1, r_2, b) with $r_1 \in [0, 1], r_2 \in [1, +\infty], b \in [0, 1]$.

A Beurling prime system exists that allows.

(*) $N_0(x) - dx = O\left(x(x)^{-1}\right)$

(**) $\inf \inf \pi_{g0}(x) \left(\frac{\log \log \log \log x}{x}\right)^{-1} = r_1$

(***) $\pi_0(x) \left(\frac{\log \log \log \log x}{x}\right)^{-1} = r_2$

In this paper we modified Beurling generalized numbers instances that demonstrate the existence of a relationship between π_0(x), $N_0(x)$ and $\zeta_0($ s) occur whenever the mathematical behavior of any one of them is known. Therefore, in this manner the body of research indicates that being aware of ψ_0(x),which leads to know the behaviors of the integers $N_0(x)$, then there's the Beurling's zeta function to consider :
where $\zeta_0(s) = \sum_{n \geq 1} n^{-s}$ where $s = \sigma +$ it, $\sigma > 1$ as follows: In the seventh decade of the last century, Diamond defined the extension of $\zeta(s)$ for a complex variable $s = \sigma +$ it, for $\sigma > 1$ in this article (see [5]) such that:

$$\zeta_0(s) = \sum_{n \geq 1} n^{-s} = \Pi_p \left(1 - \frac{1}{p^s}\right)^{-1}, \tag{3}$$

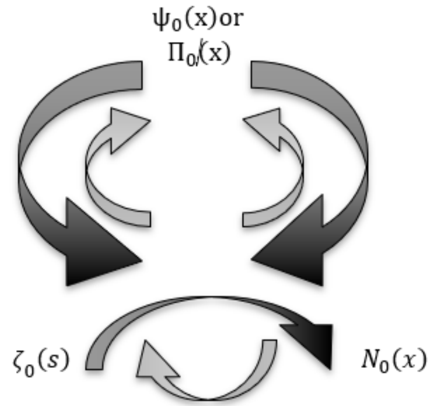

Figure 1: shows the triangular relation.

Here p is a prime number and the product of Π_p is applied all primes. It is obvious that (3) converges absolutely for $\sigma > 1$ (see[5]).

Define the space of arithmetical functions S, $f : R \to c$ such that $f(\mathrm{x})$ is right continuous and constrained local variation, $\forall x \in (1,)$.

Let $S^+ \subseteq S$ and for any f in S, f is increasing function. So, if $\mathrm{S_0} \subseteq \mathrm{S}$ (where $\mathrm{S_0} = \{\mathrm{f} \in \mathrm{S} : \mathrm{f}(1) = 0\}$), then $\mathrm{S_0}^+ = \mathrm{S} \cap \mathrm{S}^+$.

Now let's concentrate on a few easy and challenged Beurling's prime systems:
(i) Assume that K to be collection of prime numbers does not include 2. The set K ssatisfyies the of Beurling conditions of course. In other words That is; K represents Beurling's primes. Therefore, $\mathrm{N_p}$ may be shown to be the Beurling's integers "the basic theorem of arithmetic," such that $\mathrm{N_p}$ only contains only the odd numbers. In reality, this demonstrates that

$$(\pi_p(x) - 1) \sim \frac{x}{\log \log x} \quad \text{while,} \quad N_p(x) = \left[\frac{x+1}{2}\right].$$

This exactly demonstrates the existence a discrete system of Beurling's prime (π_p, N_p) in which π_p relies on K and N_p relies on the integers Beurling derives from K. Balanzario, now, define a continuous Beurling counting function $\prod_p$, according to the literature in his article (see [23]) defined a as follows:

The difficulty of generating Beurling's integers from Beurling's prime by using the Fundamental theorem of arithmetic means that knowledge of the series of generalized primes does not imply knowledge of sequence of .generalized (or Beurling) integers. The above statistic is explained by the following claims (1).

$$\zeta_{g(s)} = \int_{1-}^{\infty} x^{-s} dN_g(x) \text{ and } -\frac{\zeta_{g'(s)}}{\zeta_g(s)} = \int_{1-}^{\infty} x^{-s} d\psi_g(x)$$

Where $\pi_g(x) \sim \psi_g(x)$ for $\sigma > \frac{1}{2}$ (see [5]).
As a result of the foregoing, it is clear that :

$$\pi_p F_0^+ \left\{ \begin{array}{c} f : f \text{ is increasing function with} \\ f(1) = 0 \end{array} \right\}$$

.

$$N_p F_1^+ \left\{ \begin{array}{c} f : f \text{ is increasingfunctionwith} \\ f(1) = 1 \end{array} \right\}$$

And $N_p(x) = e^{\pi_p}$. Then the pair orders (π_p, N_p) is referred to as Beurling's prime system (see [5]).

4. liouville's function:

A part of the main objects of this work is how could we apply some of the arithmetical functions such as Liouville's function .

It's clear that appling such the mention functions is not that easy always. It needs a hard work. This function is considered one of the most important arithmetic functions in this paper, as it is used to determine the odd numbers and their relationship to this function and find its algorithm through the definition and concepts of this function. An impotent example of a completely multiplicative function is liouville's function λ which is defined as follows :

we define $\lambda(1) = 1, n = {p_k}^{a1} \ldots . . {p_k}^{ak}$ the definition then $\lambda(n) = (-1)^{a1+\cdots+ak}$ shows at once that λ is multiplicative.

Liouville's function is connected with the Riemann zeta-function by the equation:

$$\frac{\zeta(2s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s}$$

5. A Few Properties of Power Series:

The broad ideas concepts of complex half plane definitions, integral representations of Dirichlet series, and the requirements for having Euler products and simple inverses of power series are discussed here. The Liouville function $\lambda : N \to \{-1, +1\}$ is the only totally multiplicative function that equals 1 at every prime .In other words, for all primes p and natural numbers $\lambda(1) = 1$ and $\lambda(pn) = -\lambda(n)$. Alternatively, $\lambda(n) = 1^{\Omega(n)}$, where $\Omega(n)$ is the number of prime elements of n (counting multiplicity).

The Liouville function is a close relative of the Möbius function μ, and most of the material in this section has a counterpart for μ. However, we concentrate on λ for the sakeof simplicity.

Definition 5.1. $F''(x) = \dot{\chi} \cdot F_g(x) = \dot{\chi} \sum_{n=1}^{\infty} \lambda(n) \cdot \left(\text{int}\left(x^{\frac{1}{n}}\right) - 1\right)$ *Where $\dot{\chi}$ is Characteristic depending on the out come $F_g(x)$ of. So, $\dot{\chi}$ is either 1 if $F_g(x)$ is weird or 0 otherwise.*

The Arithmetical Function $F(x)$

An arithmetic function known as a Beurling function is described for the generalized prime system. It is built by employing a specific case of the generalized Chebyshev's ψ function ψ_z and might be used to generate ω-number and ω p-number positive real numbers that are divisible by two and by a power of two. The mathematical function $F(x)$ is defined in this section. It demonstrates how to obtain ω-number and ω p -number by passing a big enough real integer.

The Arithmetical Function $F(x)$

Define the function $F : S \to N$, where $S = \Big\{ | \; x \in \{x_a\}_{a \geq 1}$, such that $1 < x_1 \leq x_2 \leq x_3 \leq \cdots$ and x_a goes to ∞ as a goes to ∞ }.

The set of any real numbers belongs to a sequence x_a satisfies the conditions of Beurling's prime system (x belongs to R^+, where x is greater than a positive number a and $F(x)$ is a natural number).

Formating of $F(x)$

This work is a matter of complicated process where the function $F(x)$ is build from the Liouville function formula of Chebyshev's -function $\psi(x) = \sum_{n \geq 1} \vartheta \left(e^{\frac{\log \log x}{n}}\right)$ Where $\vartheta(x)$ is Chebyshev's ϑ-function defined by $\vartheta(x) = \sum_{q \geq x} \log \log q$ For x greater than zero and q runs over all primes less than or equal to x, then

$$\vartheta(x) = \sum_{n \geq 1} \lambda(n) \quad \psi\left(e^{\frac{\log \log x}{n}}\right)$$

Substituting Beurling's prime function $\psi_z(x)$ with $\left(\text{int}\left(x^{\frac{1}{n}}\right)-1\right)$, so

$$F(x)=\sum_{n\geq 1}\lambda(n)\left(\text{int}\left(x^{\frac{1}{n}}\right)-1\right)$$

Such that the sum is finite where n is sharply greater than $(\log\log x)(\log\log 2)^{-1}$ Where $\lambda(n)$ is the Liouville function and flow (x) is the integer part of x. If looking carefully to $F(x)$ one can see it is increasing step function with jumps 1 appears only when $x\neq a^n$ with a n greater than $1, a$ and n and are integers.

Lemma 5.2. *The summation* $\sum_{\mathrm{n}\geq 1}\lambda(\mathrm{n})\left(\text{int}\left(\mathrm{x}^{\frac{1}{\mathrm{n}}}\right)-1\right)$ *is vanished for* $n>\frac{\log\log x}{\log 2}$

Note that: Leaving the prove for the reader since it is so easy.

Theorem 5.3. *For any real* $x>e, F(\ddot{x})=\dot{\chi}\sum_{n=1}^{\infty}\lambda(n).\left(\text{int}\left(x^{\frac{1}{n}}\right)-1\right)$ *Define then* $F(x)$ *either weird number or zero where* $F(x)$ *is increasing Step function.*

Proof. In order to show the output of $F(x)$ for any large enough $x\in R$ we have to show that $F(x)$ is increasing and step function first that is to show $F(x)-F(Y)>0$. For $x,y\in R\propto x>y$.

Where let $y=x-1$ We take that $F(x)-F(Y)>0$ then

$$\begin{aligned}
&\sum_{n=1}^{\infty}\lambda(n)\cdot\left(\text{int}\left(x^{\frac{1}{n}}\right)-1\right)-\sum_{n=1}^{\infty}\lambda(n)\cdot\left(\text{int}\left(y^{\frac{1}{n}}\right)-1\right)\\
&=\sum_{n=1}^{\infty}\lambda(n)\cdot\left(\text{int}\left(x^{\frac{1}{n}}\right)-1-\text{int}\left(y^{\frac{1}{n}}\right)+1\right)\\
&=\sum_{n=1}^{\infty}\lambda(n)\cdot\left(\text{int}\left(x^{\frac{1}{n}}\right)-\text{int}\left(y^{\frac{1}{n}}\right)\right)
\end{aligned}$$

then $\left(\text{int}\left(x^{\frac{1}{n}}\right)-\text{int}\left(y^{\frac{1}{n}}\right)\right)>0\quad\forall x\in R$ therefore $\text{int}(x)\leq x\leq\text{int}(x)+1$.

Its enough to show that $\text{int}(x)-\text{int}(y)\leq 1$. Assume for the sake of argoment that for some, $n,x\in N\,\text{int}(x)>1+\text{int}(y)$

Let $p=\text{int}(y)$ then the assumption Implies $x^{\frac{1}{n}}\geq\text{int}(x)>p+1$ that is

$$x\geq(p+1)^n \tag{4}$$

However, $p\leq\text{int}\left(y^{\frac{1}{n}}\right)\leq p+1$ that is $x<(p+1)^n+1$ therefore, as both sides $x\leq(p+1)^n$ are integers this is a contradiction that with (4).

Hence $0\leq\text{int}\left(x^{\frac{1}{n}}\right)-\text{int}\left(y^{\frac{1}{n}}\right)\leq 1$. □

Theorem 5.4. *For* $f(n,x)=\text{int}\left(x^{\frac{1}{n}}\right)-\text{int}\left(y^{\frac{1}{n}}\right)$, *and* $n,x\in N$.

$$f(n,x)\begin{cases}1, & \textit{if } x=p^n, \textit{ for some } p\in N\\ 0, & \textit{if } x\neq p^n, \textit{ for any } p\in N\end{cases}$$

Proof. It's needed to show that $\text{int}\left(x^{\frac{1}{n}}\right)-\text{int}\left(y^{\frac{1}{n}}\right)\leq 1$.

Suppose for a contradiction that $\text{int}\left(x^{\frac{1}{n}}\right)>1+\text{int}\left(y^{\frac{1}{n}}\right)$ for some, $n,x\in N$.

Let $p=\text{int}\left(y^{\frac{1}{n}}\right)$ then the assumption implies $x^{\frac{1}{n}}\geq\text{int}\left(x^{\frac{1}{n}}\right)>p+1$ that is

$$x\geq(p+1)^n \tag{5}$$

However, $p \leq \text{int}\left(y^{\frac{1}{n}}\right) \leq p+1$ that is $x < (p+1)^n + 1$ therefore, as both sides $x \leq (p+1)^n$ are integers this is a contradiction that with (1).
Hence $0 \leq \text{int}\left(x^{\frac{1}{n}}\right) - \text{int}\left(y^{\frac{1}{n}}\right) \leq 1$. however $\text{int}\left(x^{\frac{1}{n}}\right)$ and $\text{int}\left(y^{\frac{1}{n}}\right)$ are integers therefore $f(n,x)$ is either 1 or 0 .
Now for $x = p^n$, for some $p \in N$.
Then $\text{int}\left(x^{\frac{1}{n}}\right) = p$ andint $\left(y^{\frac{1}{n}}\right) = \text{int}\left((p^n-1)^{\frac{1}{n}}\right) < p$.
Therefore $f(n,x) = 1$ if $x = p^n$, for some $p \in N$.
We conclude the proof by demonstrating that wherever $x \neq p^n, \quad p \in N. f(n,x) = 0$
Suppose that $\text{int}\left(x^{\frac{1}{n}}\right) - \text{int}\left(y^{\frac{1}{n}}\right) =$ such that $x \neq p^n, 1$ for some x . $p \in N\left(i \cdot \text{eint}\left(x^{\frac{1}{n}}\right) \cdot -1 = \text{int}\left(y^{\frac{1}{n}}\right)\right)$
Hence, $\text{int}\left(x^{\frac{1}{n}}\right) - 1 \leq \text{int}\left(y^{\frac{1}{n}}\right) < \text{int}\left(x^{\frac{1}{n}}\right)$. Let $p = \text{int}\left(x^{\frac{1}{n}}\right)$ we get $x^{\frac{1}{n}}$ that is,

$$x \geq p^n \tag{6}$$

However $\text{int}\left(y^{\frac{1}{n}}\right) < p$. This is $x < p^n + 1$.
Therefore as both sides are integers

$$x \leq p^n \tag{7}$$

From(2) and(3) we get $x \leq p^n, p \in N$.
This is a contradiction therefore $f(n,x) = 0$ if $x \neq p^n$ forsome $p \in N$. So it's is increasing and step obvious that $F(x)$ function.

For any real $x > e$ large enough the output of depends on $\chi(x)$ where any number $F(x)$ came from $\sum_{n=1}^{\infty} \lambda(n) \cdot \left(\text{int}\left(x^{\frac{1}{n}}\right) - 1\right)$ should be either weird or zero by the affection of $\dot{\chi}(x)$. □

Results, Discussion and future works

It is possible to add and modify the basic function in this work, as it is considered among the theory's most significant arithmetic functions of numbers. Moreover, it has sufficient security in terms of application in the life aspect, as well as in this work there is an applied aspect and a mathematical algorithm that is in line with the nature and work of the research.

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