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Numerical identification of timewise dependent coefficient in Hyperbolic inverse problem

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Abstract This article investigates the nonlocal inverse initial boundary-value problem in a rectangular domain, hyperbolic second order inverse problem. The main objective is to find the unidentified coefficient and offer a solution to the problem. The hyperbolic second-order, nonlinear equation is solved using finite difference method (FDM). However, the inverse problem was successfully solved by the MATLAB subroutine lsqnonlin from the optimization toolbox after being reformulated as a nonlinear regularized least-square op-timization problem with a simple bound on the unknown quantity. Given that the studied problem is often ill-posed and that even a minor error in the input data can have a large impact on the output. Tikhonov's regularization technique is used to generate stable and accurate results.

Keywords:. Finite difference method, Tikhonov regularization method, hyperbolic inverse problem, Inverse problem.

1. Introduction

Partial differential equations play an essential role in many areas of science and enginearing. For example, in engineering, design, construction, and medicine, numerical solutions to partial differential equations are more developed and modern than analytical methods, especially after the advent of high-speed computing machines. Where the development of the numerical method was strongly affected and this progress is still being seen constantly. Inverse problems in phoneme setting have been examined by many authors such Radial Basis Functions method and Fragile Points Method applied in [10] and [33] respectively, Tikhonov regularization method used in [14], [29], [6], [13], [20], [2], [16] and Lavrentiev regularization method [21]. The twentieth century is constantly being researched for practical applications, such as in medicine, biology geophysics mineral investigation, filtration and computer tomography [12], [24], [31].

Yashar et al. in [27] presented hyperbolic inverse problem from higher order to prove the exitances and uniqueness then they identified the unknown time depended coefficients. Aysel and Yashar in 2020 [30] established the existence and uniqueness of hyperbolic inverse problem for fourth order to determining the lowest coefficient. In [25], Yashar et al, studied the uniqueness and existence for the equation of flexural vibrations of a bar with nonlocal integral conditions. Yusif et al, in [26] presented the uniqueness and existence of fourth order hyperbolic equation with periodic conditions.

The solution of hyperbolic problems has great interest, it has been studied by many researchers such as, in [20] the approximate solution of the inverse hyperbolic problem with the left end flux tension of the string, by using the finite difference method. The numerical solution of the inverse problem for the hyperbolic equation with overdetermination is accomplished to identify both space and time-dependent forcing term in [5]. The author in [7] identification the time coefficient of hyperbolic second order inverse problem with variable coefficients.

In [3], [32] the hyperbolic inverse problem are studied to investigated the time- dependent coefficients with different type of boundary conditions. Also, Eskin in [8] studied second order hyperbolic inverse problem to identify the time -dependent coefficient. The author in [22] studied the hyperbolic heat conduction inverse problem to reconstruct the unknown surface heat flux from the temperature measurements. While, in [1] presented the classical solution for the linearized equation of motion of a homogeneous elastic beam which is hyperbolic fourth order problem with periodic conditions.

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In this study, a one- dimensional hyperbolic inverse problem was presented of the second- order to investigate the retrieval of timewise potential term numerically from the additional measurement, with initial conditions and non-local integral condition. The uniqueness and existence already proved by Yashar in [1], but no numerical investigation has been carried out till now.

The paper is structured as follows: The mathematical formulation of the problem is presented in Section 2. Section 3 describes the direct finite difference scheme for obtaining the numerical solution to a direct problem with stability analysis, along with numerical test example. The inverse problem presented in Section 4. Section 5 presented the numerical results of inverse problem, and the conclusion of the paper is shown in section 6.

2. The formultion of the problem

Let $Q_T = \{0 \leq x \leq 1, 0 \leq t \leq T\}$ be a rectangle domain. Consider the following inverse problem of determining a pair of functions $(u(x,t), p(t))$, [1]

$$u_{tt} - u_{xx} = p(t)u + f(x,t), \tag{1}$$

the initial condition

$$u(x,0) + s_1 u(x,T) = \varphi(x), u_t(x,0) + s_2 u(x,T) = \psi(x) \quad 0 \leq x \leq 1, \tag{2}$$

the periodic boundary condition:

$$u(0,t) = \lambda u(1,t), \quad 0 \leq t \leq T, \tag{3}$$

the non-local condition

$$\int_0^1 u(x,t)dx = 0, \quad 0 \leq t \leq T, \tag{4}$$

and the final overdetermination condition

$$u(1/2,t) = h(t), \quad 0 \leq t \leq . \tag{5}$$

We call the equations 1 - 5, the inverse problem. The functions $f, \Phi, \psi, s_1, s_2, \lambda$ and h are given functions, where $s_1, s_2 \geqslant 0$ and $\lambda \neq \pm 1$. In this problem $p(t)$ is represents the potential term, and $u(x,t)$ represents the temperature distribution over the rectangle domain at position x and time t and these functions are unknown.

2.1. Existence of the classical solution for inverse problem

Definition 2.1. *A pair of functions $u(x,t), p(t)$ is said to be a classical solution of problem 1-5 if the following conditions is satisfied:*
i: $u(x,t), u_{xx}(t)$ and $u_{tt}(x,t) \in C[0,T]$ in Q_T.
ii: $p(t) \in C[0,T]$.
iii: The Eqs. 1 –5 are satisfied.
Now to study problem 1-5 we consider the auxiliary inverse problem as

$$u_x(0,t) = u_x(1,t), \quad 0 \leq t \leq T \tag{6}$$

$$h''(t) - u_{xx}(1/2,t) = p(t)h(t) + f(1/2,t), \quad 0 \leq t \leq T. \tag{7}$$

Theorem 2.2. *Assume* $\varphi(x), \psi(x) \in C[0,1], f(x,t) \in C(Q_T), \int_0^1 f(x,t)dx = 0, h(t) \in C^2[0,T], h(t) \neq 0 (0 \leq t \leq T)$, *and the following compatibility conditions are hold:*

$$\int_0^1 \varphi(x)dx = 0, \quad \int \psi(x)dx = 0 \tag{8}$$

$$\varphi(1/2) = h(0) + s_1 h(T), \quad \psi(1/2) = h'(0) + s_2 h'(T) \tag{9}$$

Then the following is hold:

1. *Each classical solution of the inverse boundary value problem 1-5 is the solution of problem 1-3, 6, 7.*
2. *Each solution of the inverse boundary value problem 1-3, 6, 7 is a classical solution of problem 1-5 if*

$$\frac{(1 + 2s_1 + 3s_2 + s_1 s_2)T^2}{2(1 + s_1)(1 + s_2)} < 1. \tag{10}$$

Lemma 2.3. *Let the input data of problem 1–5, satisfy the following conditions:*

1. $s_1, s2 \geqslant 0, \quad 1 + s_1 s_2 \geqslant s_1 + s_2$
2. $\varphi(x) \in C^2[0,T], \varphi^{'''}(x) \in L_2(0,1) \varphi(0) = \lambda\varphi(1), \varphi^{'}(0) = \varphi^{'}(1), \varphi^{''}(0) = \lambda\varphi^{''}(1)$
3. $\psi(x) \in C^1[0,T], \psi^{''}(x) \in L_2(0,1), \psi(0) = \lambda\psi(1), \psi^{'}(0) = \psi^{'}(1).$
4. $f(x,t), f_x(x,t), f_{xx}(x,t) \in C(Q_T), f(0,t) = \lambda f(1,t), f_x(0,t) = f_x(1,t), \quad 0 \leq t \leq T$
5. $h(t) \in C^2[0,T], \quad h(t) \neq 0, \quad 0 \leq t \leq T$

Theorem 2.4. *Let the conditions 1-5 be satisfied and*
$(E(T) + 2)^2 L(T) < 1.$
Then problem 1– 5, has a unique solution in $K = K_R(\|z\|_{E_T^5} \leq R = E(T) + 2))$ *in the space* E_T^5. *Where*

$$E(T) = E_1(T) + E_2(T) + E_3(T) + E_4(T)$$

$$L(T) = L_1(T) + L_2(T) + L_3(T) + L_4(T)$$

$$E_1(T) = \frac{2}{1+s_1}\|\varphi(x)\|_{L_2(0,1)} + \frac{2T}{1+s_2}\|\psi(x)\|_{L_2(0,1)} + \frac{2(1+3s_1+3s_2)}{(1+s_1)(1+s_2)} T\sqrt{T}\|f(x,t)\|_{L_2(Q_T)}.$$

$$L_1(T) = \frac{(1+3s_1+3s_2)}{((1+s_1)(1+s_2))T^2},$$

$$E_2(T) = 4\sqrt{2}r(1+s_2)\|\varphi^3(x)\|_{L_2(0,1)} + 4\sqrt{2}r(1+s_1)\|\psi^2(x)\|_{L_2(0,1)} + 4(1+2r(s_1+s_2+s_1 s_2))\sqrt{2T}\|f_{xx}(x,t)\|_{L_2(Q_T)},$$

$$L_2(T) = 2(1 + 2r(s_1 + s_2 + s_1 s_2))T,$$

$$E_3(T) = 8r(1+s_2)\|\varphi^3(x)(1-g-rx) - 3r\varphi^2(x)\|_{L_2(0,1)} + 8r(1+s_1)\|\psi^2(x)(1-g-rx) - 2r\psi^{(1)}(x)\|_{L_2(0,1)} + 8(1+2r(s_1+s_2+s_1 s_2))\sqrt{T}\|f_{xx}(x,t)(1-g-rx) - 2rf_x(x,t)\|_{L_2(Q_T)} + 8r_1\|\varphi^{(3)}(x)\|_{L_2(0,1)} + 8r_2\|\psi^{(2)}(x)\|_{L_2(0,1)} + 8(1+2r(s_1+s_2+s_1 s_2))^2 T\sqrt{T}\|f_{xx}(x,t)\|_{L_2(Q_T)}.$$

$$L_3(T) = 2\sqrt{2}(1 + 2r(s_1 + s_2 + s_1 s_2))T + 2\sqrt{2}(1 + 2r(s_1 + s_2 + s_1 s_2))^2 T^2,$$

$$
\begin{aligned}
E_4(T) = \left\|[h(t)]^{-1}\right\|_{C[0,T]} \Bigg\{ & \left\| h''(t) - f\left(\frac{1}{2}, t\right) \right\|_{C[0,T]} \\
& + \frac{1}{2}\left(\sum_{k=1}^{\infty} \zeta_k^{-2}\right)^{\frac{1}{2}} \Big[2\sqrt{2}r\,(1+s_2) \left\|\varphi^{(3)}(x)\right\|_{L_2(0,1)} \\
& + 2\sqrt{2}r\,(1+s_1) \left\|\psi^{(2)}(x)\right\|_{L_2(0,1)} \\
& + (1 + 2r\,(s_1+s_2+s_1s_2))\, 2\sqrt{2T}\, \|f_{xx}(x,t)\|_{L_2(Q_T)} \Big] \Bigg\},
\end{aligned}
$$

$$
L_4(T) = \frac{1}{2}\left\|[h(t)]^{-1}\right\|_{C[0,T]} \left(\sum_{k=1}^{\infty}\zeta_k^{-2}\right)^{\frac{1}{2}} \left[(1+2r\,(s_1+s_2+s_1s_2))\,T\right].
$$

Theorem 2.5. *Let all the conditions of Theorem 1 be satisfied, and*

$$
\int_0^1 f(x,t)dx = 0, \qquad (0 \le t \le T),
$$

and the compatibility conditions are met:

$$
\int_0^1 \varphi(x)dx = 0, \quad \int_0^1 \psi(x)dx = 0, \quad \varphi\left(\frac{1}{2}\right) = h(0) + \mathrm{sh}(T),
$$

$$
\psi\left(\frac{1}{2}\right) = h'(0) + s'(T),
$$

$$
\frac{(1+2s_1+3s_2+s_1s_2)\,T^2(E(T)+2)}{2\,(1+s_1)\,(1+s_2)}.
$$

Then the inverse problem (1)-(5) has a classical solution in the ball $K = K_R\left(\|z\|_{E_T^5} \le R = E(T)+2\right)$ from E_T^5, for proof see [23].

3. Discretization of the direct solver

Consider the direct solver for the inverse problem contain the equations (1)-(4) and required output data (5). In this direct problem the only unknown quantity that should be determine is $u(x,t)$ that is all other components are given. Discretizing Eq. (1) by a form of (FDM) as follows [14], [11],[9], [18], [28]: Denote for $u(x_i,t_j) = u_{i,j}$, and $f(x_i,t_j) = f_{i,j}$ where space node $x_i = i\Delta x$, time node $t_j = j\Delta t$, the space step length $\Delta x = \frac{1}{M}$ and time step length $\Delta t = \frac{T}{N}$ for $i = 0,1,\ldots,M, j = 0,1,2,\ldots,N$ where M,N are positive integers. Based on the FDM scheme Crank-Nicolson scheme, Eq. (1) can be expressed as:

$$
\begin{aligned}
\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta t^2} = & \left(\frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}}{2(\Delta x)^2} + \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{2(\Delta x)^2}\right) \\
& + \frac{1}{2}p_{j+1}u_{ij+1} + \frac{1}{2}p_j u_{ij} + \frac{1}{2}\left(f_{i,j+1} + f_{i,j}\right), \\
& i = 1,2,\ldots,M,\ j = 0,1,\ldots,N,
\end{aligned} \tag{11}
$$

simplifying the last equation, we have:

$$
\begin{aligned}
u_{i,j+1} - 2u_{i,j} + u_{i,j-1} = & \Delta t^2 \left(\frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}}{2(\Delta x)^2} + \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{2(\Delta x)^2}\right) \\
& + \frac{\Delta t^2}{2}p_{j+1}u_{ij+1} + \frac{\Delta t^2}{2}p_j u_{ij} + \frac{\Delta t^2}{2}\left(f_{i,j+1} + f_{i,j}\right),
\end{aligned} \tag{12}
$$

$$A = \frac{\Delta t^2}{2(\Delta x)^2}, \qquad B = \frac{\Delta t^2}{2} p_{j+1}.$$

Let $s_1 = s_2 = 0$ for simplicity then the initial conditions be:

$$u(x_i, 0) = \varphi(x_i), \quad u_t(x_i, T) = \psi(x_i) \quad i = 1, \ldots, M, \tag{13}$$

$$u(0, t_j) = \lambda u(1, t_j), \quad j = 0, 1, \ldots, N, \tag{14}$$

can be approximated as:

$$u_{0,j} = \lambda u_{M,j}, \quad \text{for all} \quad j = 0, 1, \ldots, N,$$

and the second periodic condition gives,

$$\frac{u_{j+1} - u_{j-1}}{2\Delta t} = \psi(x),$$
$$u_{j-1} = u_{j+1} - 2\Delta t \psi(x) \quad \text{for all} \quad j = 0, 1, 2, \ldots, N.$$

Using the trapezoidal rule to approximate the integral (4) to reach the following expression,

$$u_{0j} + 2\sum_{i=1}^{M-1} u_{ij} + u_{Mj} = 0, \quad j = 0, 1, \ldots, N. \tag{15}$$

And the overdetermination condition (5) is approximated as:

$$h(t_j) = u_{\frac{1}{2},j}, \quad j = 0, 1, 2, \ldots, N$$

Then (12) can be rearranged into the following difference equation

$$\begin{aligned} -Au_{i-1,j+1} + &(1 + 2A - B)u_{i,j+1} - Au_{i+1,j+1} \\ &= Au_{i-1,j} + (2 - 2A + B)u_{i,j} + Au_{i+1,j} - u_{i,j-1} + \frac{\Delta t^2}{2}\left(f_{i,j+1} + f_{i,j}\right), \\ &i = 1, 2, 3, \ldots, M, \ j = 0, 1, \ldots, N. \end{aligned} \tag{16}$$

The last difference equation can be expressed in a more convenient way as the following linear algebraic system

$$\boldsymbol{D} v^{j+1} = \boldsymbol{E} v^j + Z,$$

where the matrices D and E have the following form A time node t_1, we have

$$D = \begin{pmatrix} 1.5 & 2 & 2 & \cdots & 2 & 2 & 2 \\ -A & 2 + 2A - B_j & -A & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & -A & \cdots & -A & 2 + 2A - B_j & -A \\ -\frac{A}{2} & 0 & 0 & \cdots & 0 & -A & 2 + 2A - B_j \end{pmatrix}_{M \times M},$$

$$E = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ A & 2 - 2A + B_j & A & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A & 2 - 2A + B_j & A \\ A & & 0 & \cdots & 0 & A & 2 - 2A + B_j \end{pmatrix}_{M \times M},$$

$$Z = 2\Delta t * \begin{pmatrix} 0 \\ \psi_{1,j} \\ \psi_{2,j} \\ \vdots \\ \psi_{M-1,j} \end{pmatrix} + \frac{\Delta t^2}{2} \begin{pmatrix} 0 \\ f_{1,j+1} + f_{1,j} \\ f_{2,j+1} + f_{2,j} \\ \vdots \\ f_{M-1,j+1} + f_{M-1,j} \end{pmatrix}.$$

For the rest of time nodes $t_2, t_3, \ldots, t_N$, we have:

$$D = \begin{pmatrix} 1.5 & 2 & 2 & \cdots & 2 & 2 & 2 \\ -A & 1+2A-B_j & -A & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & -A & \cdots & -A & 1+2A-B_j & -A \\ -\frac{A}{2} & 0 & 0 & \cdots & 0 & -A & 1+2A-B_j \end{pmatrix}_{M\times M},$$

$$E = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ A & 2-2A+B_j & A & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A & 2-2A+B_j & A \\ \frac{A}{2} & 0 & 0 & \cdots & 0 & A & 2-2A+B_j \end{pmatrix}_{M\times M},$$

$$Z = -\begin{pmatrix} 0 \\ u_{1,j} \\ u_{2,j} \\ \vdots \\ u_{M-1,j} \end{pmatrix} + \frac{\Delta t^2}{2}\begin{pmatrix} 0 \\ f_{1,j+1}+f_{1,j} \\ f_{2,j+1}+f_{2,j} \\ \vdots \\ f_{M-1,j+1}+f_{M-1,j} \end{pmatrix},$$

where, $v^{j+1} = (u_{0,j+1}, u_{1,j+1}, \ldots, u_{M-1,j+1})$ and $v^j = (u_{0,j}, u_{1,j}, \ldots, u_{M-1,j})$.

3.1. Stability analysis of direct problem

In this subsection, we apply the Von Neumann stability analysis for direct problems (1)-(4) [13], [34]. We take $f(x,t) = 0$, for simplicity, and assuming local constant $p_j = \hat{g}$ for known level in Eq. (16) where $\hat{g} = \max_{t=[0,T]} |p(t)|$, then the difference equation becomes:

$$\begin{aligned} -Au_{i-1,j+1} + &(1+2A-B)u_{i,j+1} - Au_{i+1,j+1} \\ &= Au_{i-1,j} + (2-2A+B)u_{i,j} + Au_{i+1,j} - u_{i,j-1}, \\ &\quad i = 1,2,3,\ldots,M,\ j = 0,1,\ldots,N \end{aligned} \tag{17}$$

$$A = \frac{\Delta t^2}{2(\Delta x)^2}, \qquad B = \frac{\Delta t^2}{2}\hat{g}$$

applying decomposition method of the numerical solution into a Fourier sum as

$$u_{i,j} = S^j e^{wi\theta}, \tag{18}$$

where S is the amplification factor, the phase angle $\theta = \emptyset \Delta x$, where $\emptyset = \frac{2\pi}{N}$ and $w = \sqrt{-1}$ and Δx is the space length. If |S| < 1, then we said S to be satisfying the von Neumann condition and the FDM scheme is stable. To find S, substitute the above data into Eq. (17) as follows:

$$(-2A\cos\theta + (1+2A-B))S^2 - (2A\cos\theta + (2+2A+B))S + 1 = 0,$$

we can be written as

$$\gamma_1 S^2 - \gamma_2 S + \gamma_3 = 0, \tag{19}$$

where

$$\gamma_1 = -2A\cos\theta + (1+2A-B), \quad \gamma_2 = 2A\cos\theta + (2+2A+B), \quad \gamma_3 = 1,$$

under the transformation $S = \frac{1+\omega}{1-\omega}$ in (18) then we get

$$(\gamma_1+\gamma_2+\gamma_3)\,\omega^2 + 2\,(\gamma_1-\gamma_3)\,\omega + (\gamma_1-\gamma_2+\gamma_3) = 0,$$

the discretized system (17) will be stable if

$$\gamma_1+\gamma_2+\gamma_3 \geq 0, \quad \gamma_1-\gamma_3 \geq 0, \quad \gamma_1-\gamma_2+\gamma_3 \geq 0,$$

after simplifying the terms above, we get

$$\gamma_1+\gamma_2+\gamma_3=\frac{2\Delta t^2}{(\Delta x)^2}+3, \tag{20}$$

$$\gamma_1-\gamma_3=\frac{\Delta t^2}{(\Delta x)^2}\left(2\sin^2\frac{\theta}{2}-\Delta x^2\hat{g}\right), \tag{21}$$

$$\gamma_1-\gamma_2+\gamma_3=2+\frac{2\Delta t^2}{(\Delta x)^2}\left(2\sin^2\frac{\theta}{2}-\frac{\Delta x^2}{2}\hat{g}-1\right). \tag{22}$$

It is clear from (18) that $\gamma_1+\gamma_2+\gamma_3 \geq 0$. From (21) and (22), we get, $\gamma_1-\gamma_3 \geq 0$ and, $\gamma_1-\gamma_2+\gamma_3 \geq 0$ if $(\Delta x)^2 \leq \frac{1}{\hat{g}}\left(2\sin^2\frac{\theta}{2}-1\right)$. That is, the proposed method will be stable.

The convergence of the proposed scheme is obtained directly from the Lax-Richtmyer equivalence theorem states that "a consistent finite-difference scheme for a partial differential equation for which the initial-value problem is well posed is convergent if and only if it is stable", see [35].

3.2. *Example for direct problem*

Consider the direct problem (1)-(4) with T = 1, and the following input data:

$$\begin{aligned}
\varphi(x) &= \sin(2\pi x), \quad x \in [0,1], \\
\psi(x) &= -\sin(2\pi x), \quad x \in [0,1], \\
p(t) &= \mathrm{e}^{\sqrt{10000t}}, \quad t \in [0,T], \\
f(x,t) &= -e^{-t}\left(-1+\mathrm{e}^{\sqrt{10000t}}-4\pi^2\right)\sin(2\pi x), \quad (x,t) \in Q_T.
\end{aligned}$$

the analytic solution is given by

$$u(x,t)=e^{-t}\sin(2\pi x), \quad (x,t) \in Q_T$$

and overdetermination condition

$$h(t)=e^{-t}+1.22465*10^{-16}, \quad t \in [0,T]$$

this solution can be verified by direct substitution into governing equation. The numerical and analytical results for the temperature distribution $u(x,t)$ at coarse mesh size $M_=N_=40$, is depicted in Figure 1 and very good accuracy is obtained as illustrated in absolute error graph which about 10^{-8} magnitude, see right hand plot. Figure 2 displays the computational required data in comparison with the analytical one for $h(t)$ for $s_1=s_2=0 and \lambda=2$, and excellent agreement is also obtained.

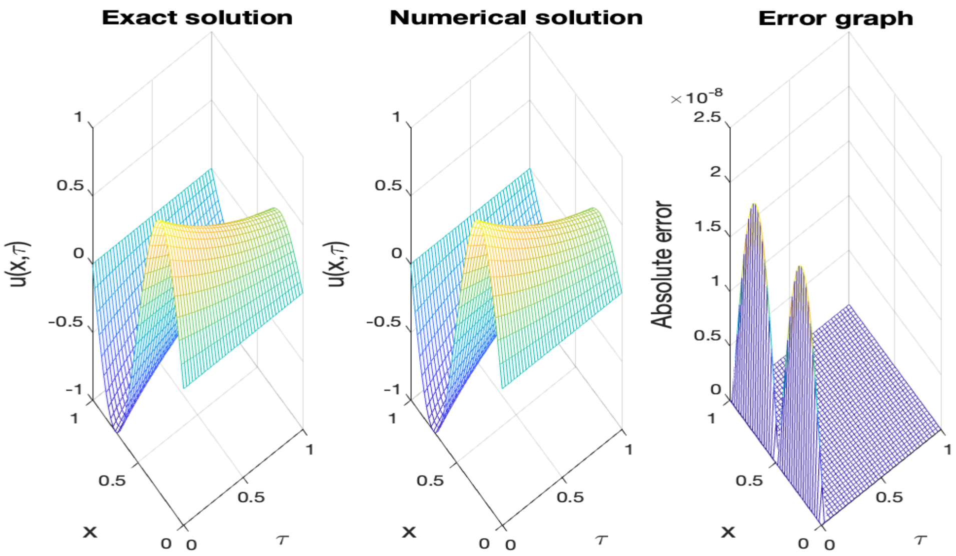


Figure 1: Analytical and computational temperature distributions for $u(x,t)$ and the absolute error between them.

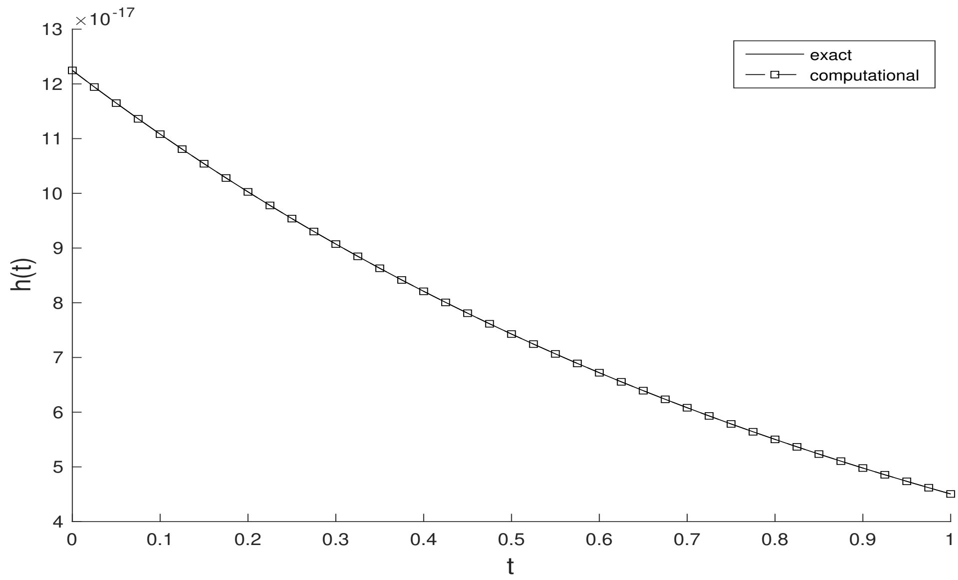


Figure 2: The analytical and computational curve for $h(t)$ for the forward (direct) problem.

4. Inverse Problem

Our goal in this section is devoted for solving the inverse problem. To find stable reconstructions for unknown coefficient p(t), in additional to heat distribution $u(x,t)$ which satisfy Eqs. 1- 5. This problem is solved numerically by minimizing the gap between extra measurement data 5 and computed solution. To gain suitable results we apply the Tikhonov's regularization method due to ill-posedness of the problem. The cost functional can be constructed from 5 for more details see [15], [17], [4], [19];

$$K(p) = \|u(1/2,t) - h(t)\|^2 + \beta\|p(t)\|^2, \tag{23}$$

and the approximate formula is

$$K(\underline{p}) = \sum_{j=1}^{N} (u(1/2, t_j) - h(t_j))^2 + \beta \sum_{j=1}^{N} p_j^2, \tag{24}$$

where$\beta \geqslant 0$ the regularization parameter, and the norm is the usual norm over $[0, T]$: The objective function (23), it is minimized by subroutine *lsqnonlin* from MATLAB optimization toolbox. This routine try to solve nonlinear least- squares curve fitting problem starting from the initial guess. The upper and lower bounds on the variable p are specified as $10^(- 2) \leq p \leq 10^2$ for Example1 and $-200 \leq p \leq 200$ for Example 2. Also in this routine, is not required the gradient to be supplied by the user which is computed inside the routine via some FDM formula.

The following parameters are essential to start optimization processes of (24), the minimization will terminate when the following prescribed parameters are achieved:

1. Allowed number of iterations= 50∗(No. of variables).

2. Specified solution and objective function Tolerance $= 10^{-20}$.

The inverse problem is solved with respect to noisy/ exact measurement data in (5). The additive noise as presented in:

$$h^{\epsilon}(t_j) = h(t_j) + \epsilon_j, \quad j = 1, 2, \ldots, N, \tag{25}$$

where ϵ is a normal Gaussian random vector with zero mean and standard deviation μ is:

$$\mu = q \times \max_{t \in [0,T]} |h(t)|, \tag{26}$$

where q represents the percentage of noise. Here we use the normrnd built-in function to generate the random variables $\epsilon = (\epsilon_j) \quad j = 1, 2, \ldots, N$ as follows:

$$\epsilon = normrnd(0, \mu, N). \tag{27}$$

5. Results and discussion

We introduce couple of test examples for inverse problem. To explain and validate the stability and accuracy of the computational procedure which is based on finite difference method combined with the minimization of functional (23).

5.1. Example 1:

Consider data for inverse problem Eqs (1)-(5) as follows:

$$u(x,t) = \frac{\sin(\frac{t}{6})\cos(2x\pi)}{100}, \quad (x,t) \in Q_T$$

and the input data are as follows:

$$f(x,t) = (0.424506 + 0.942478t)\cos(2\pi x)\sin(\frac{t}{6}), \quad (x,t) \in Q_T$$

$$\varphi(x) = 0, \quad \psi(x) = (\cos(2\pi x))/600, \quad h(t) = -\sin(t/6)/100, \quad x \in [0,1], \quad t \in [0,1],$$

$$p(t) = -3 - 30\pi t, \quad t \in [0,1]$$

The initial guess was $p_0 = -3$. It is easy to verify the input data for the conditions of the Theorems 1-3 except that $h(0) = 0$. Hence, the inverse hyperbolic problem (1)-(5) with input data above may has a unique solution. We fix $M = N = 40$ for the numerical investigation started with the situation of no noise included, i.e., $q = 0$. The objective function (23) represented in Figure 4(a), and a speed declining convergence in the first 10 iterations is seen for achieving a shorter order tolerance $O(10^{-15})$ in 50 iterations. Figure 3 shows numerical results for the coefficient $p(t)$ and it is clearly good results with reasonable amount of accuracy except a small unstabization appears at the end of interval.

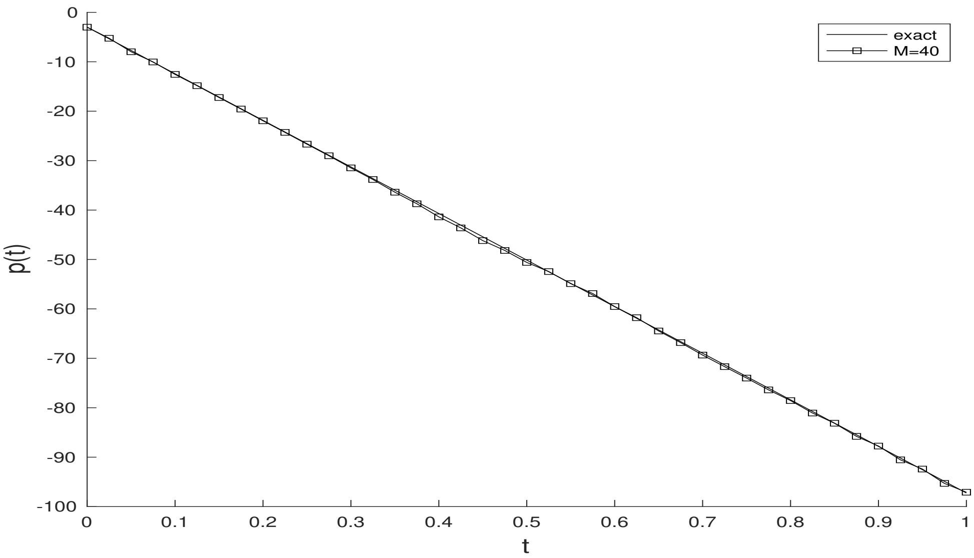
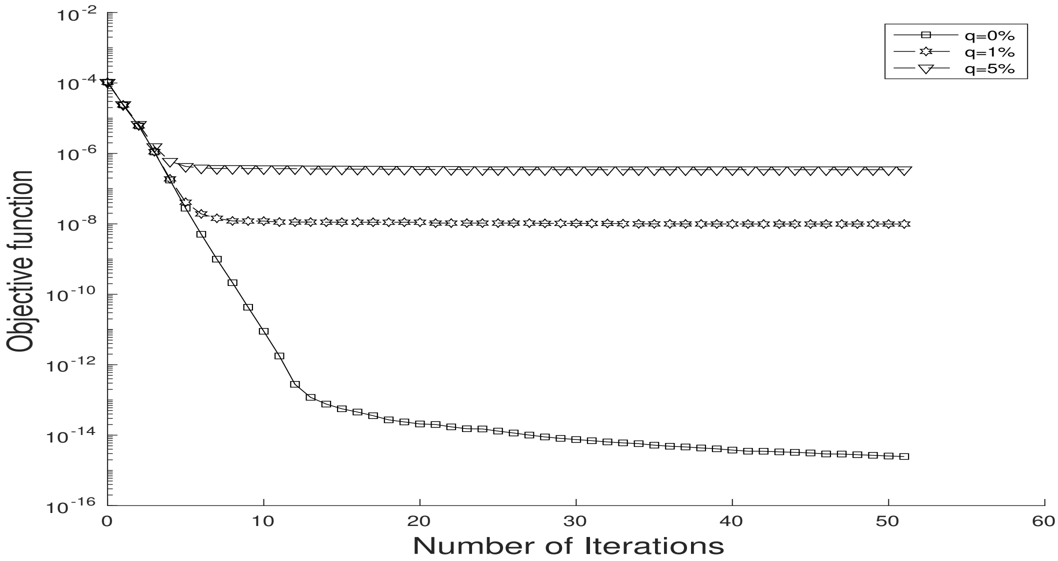
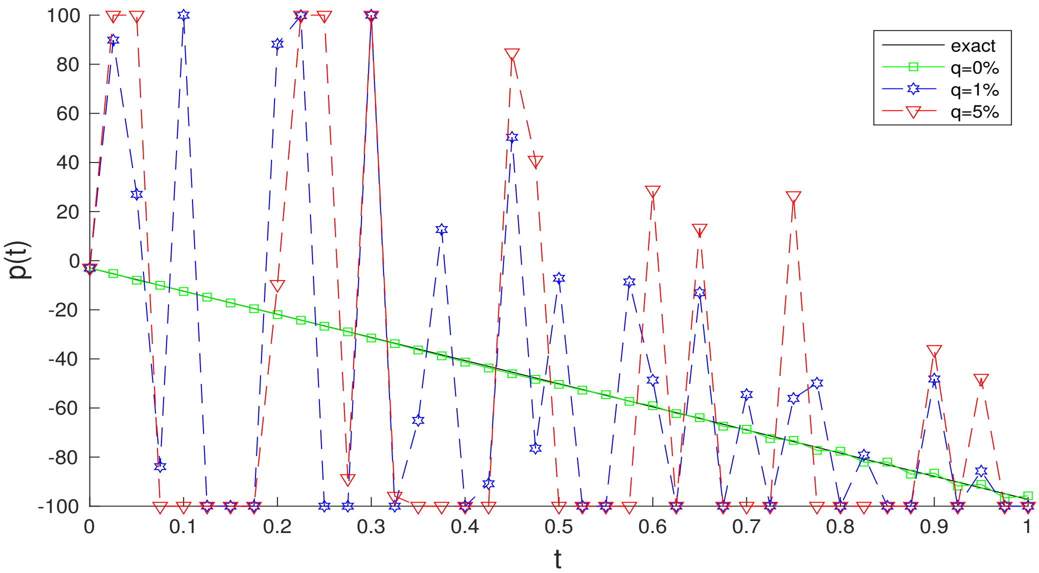


Figure 3: $p(t)$ with noise free and without regularization.

Next, we contaminate the input data with $q \in \{1, 5\}$ noise as in equation (26). The case of noisy data and no regularization is presented in Figures 4(a)-4(b). Figure 4(a) show convergence of minimization processes due to inclusion of noise and absence of regularization to reach stationary values of order $O(10^{-7})$ and $O(10^{-6})$ for $q \in \{1, 5\}$ respectively.



(a)

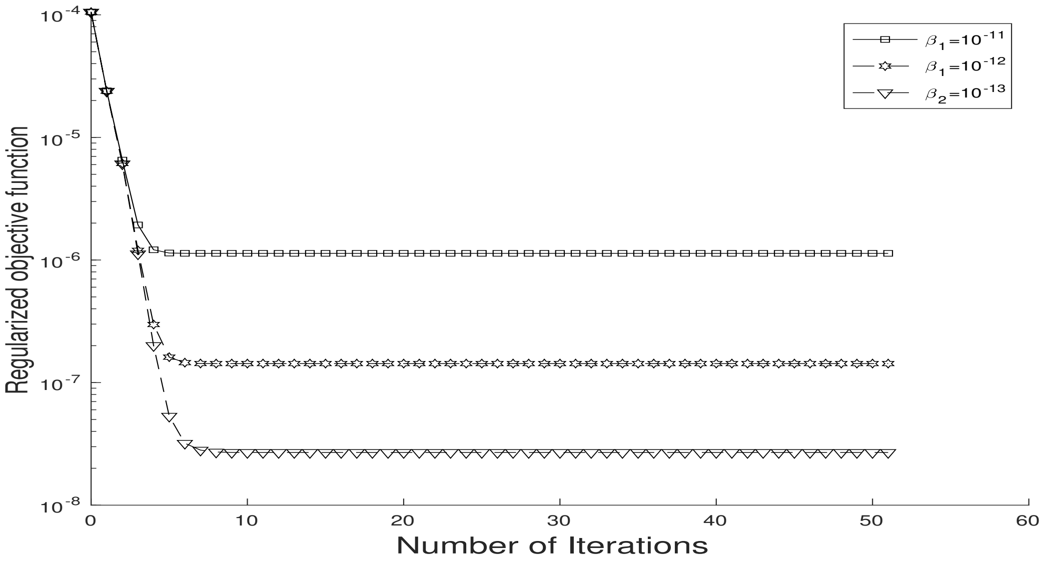


(b)

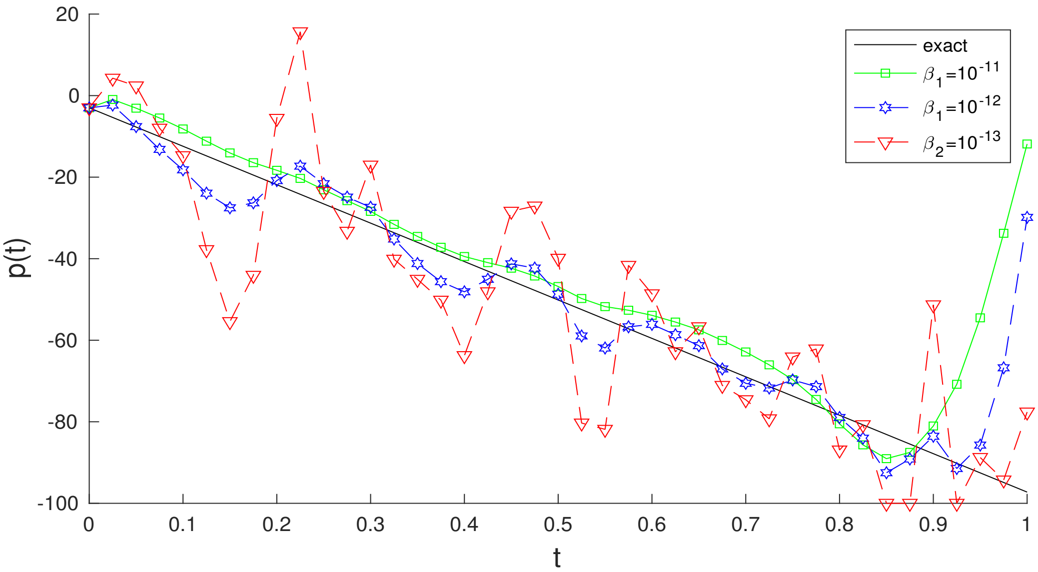
Figure 4: (a) objective function (23) and (b) $p(t)$, for different noise level $q \in \{1, 5\}$ and no regularization.

This is expected since the problem under investigation is ill-posed problem and small errors (noise) in input data lead to drastic errors in outputs. As seen in Figure 4(b), the numerical solution of the $p(t)$ stable and accurate at $q = 0$ and when $q = 1$, and with $q = 5$, unstable and diverges from the exact solution but remains on the same path when the value of additive noise increases.

By incorporating the penalty terms $\beta\|p(t)\|^2$ into equation (23), we employ the Tikhonov regularization technique. We try out different values for the regularization parameter $\beta \in \{10^{-13}, 10^{-12}, 10^{-11}\}$, noise of $q = 1$ is added to replicate real input data. In Figures 5(a) and 6(a), the monotonic decreasing achieved in about 50 iterations and noise of $q = \{1, 5\}$, indicating that the objective function minimization (23) is satisfied. Figures 5(b) and 6(b) depict the unknown potential coefficient $p(t)$. The noise levels as increased from 1 to 5 the instabilities appear.

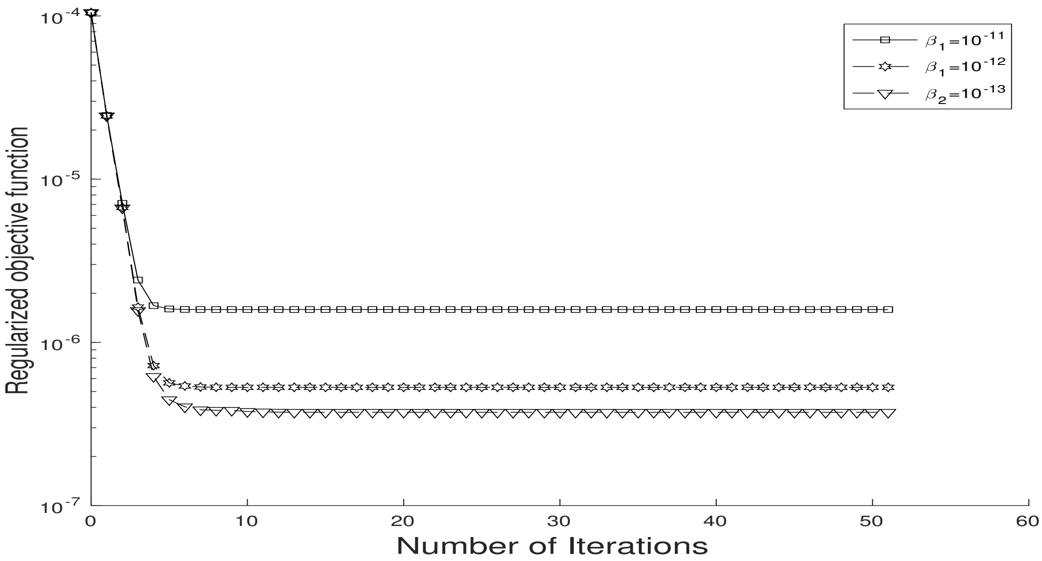


(a)

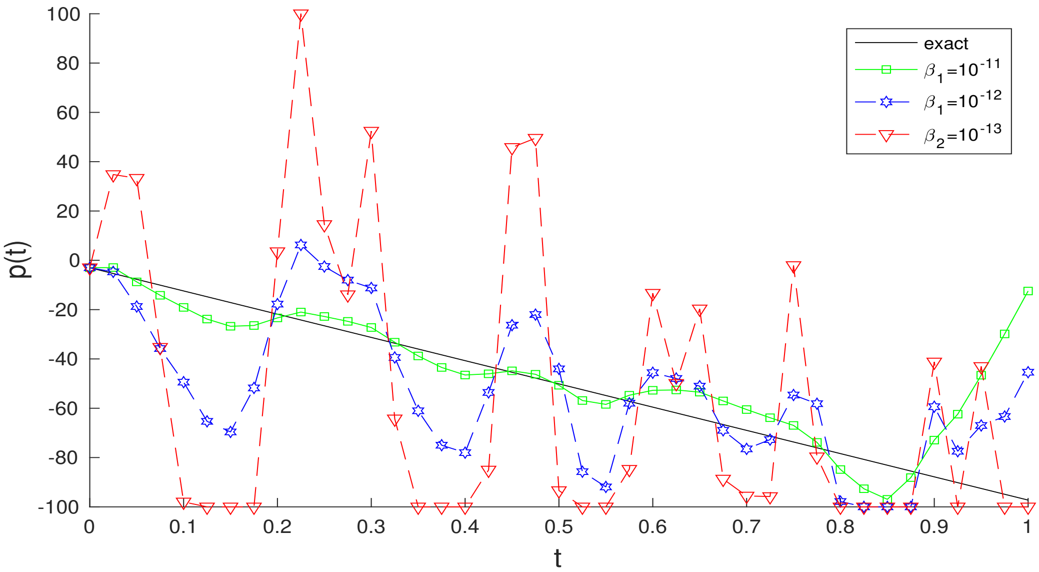


(b)

Figure 5: (a) objective function (23) and (b) $p(t)$, for $q = 1$ noise and $\beta \in \{10^{-13}, 10^{-12}, 10^{-11}\}$.



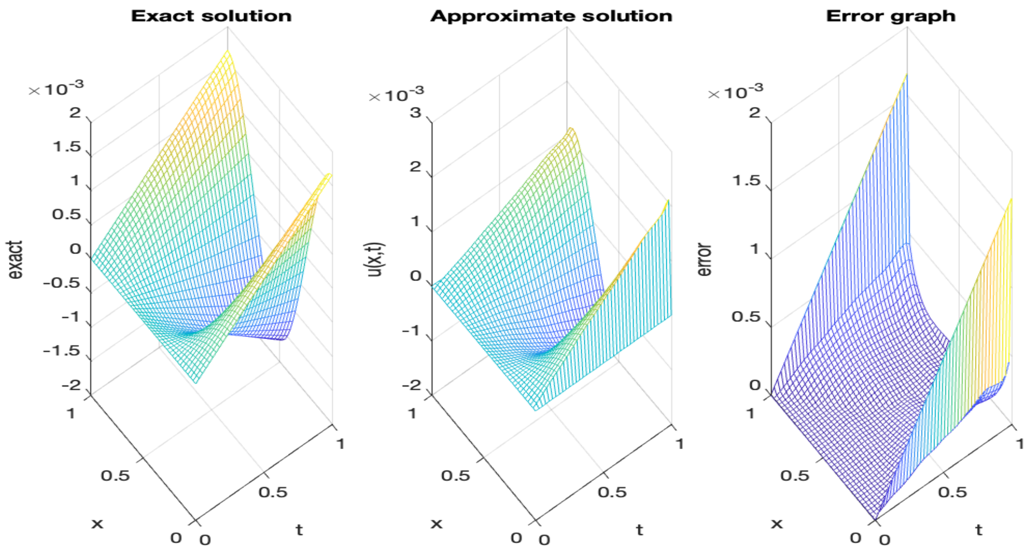
(a)



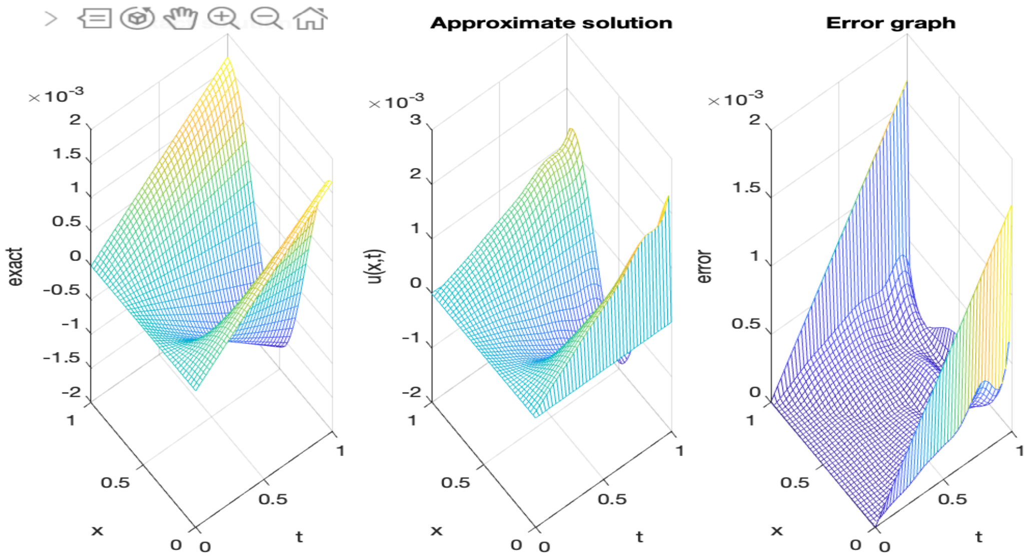
(b)

Figure 6: (a) objective function (23) and (b) $p(t)$, for $q = 5$ noise and $\beta \in \{10^{-13}, 10^{-12}, 10^{-11}\}$.

The numerical and exact temperatures $u(x,t)$, with $q = 1$noise, $\beta = \{10^{-12}\}, q = 5$ noise, $\beta = \{10^{-11}\}$, as well as the absolute error between them, are illustrated in Figure 7 and execute arguments obtained.



(a)



(b)

Figure 7: The exact and numerical $u(x,t)$ with (a) $q = 1\%$noise, $\beta = \{10^{-12}\}, q = 5\%$noise, $\beta = \{10^{-11}\}$, as well as, the absolute error between them.

5.2. Example 2:

Consider data of inverse problem Eqs (1)-(5) as follows:

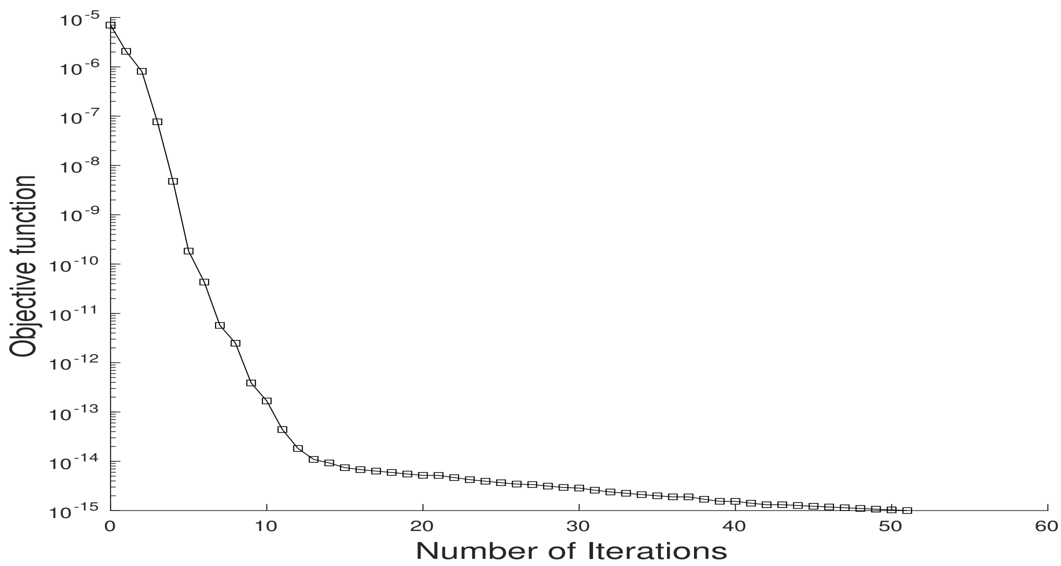
$u(x,t) = (\cos(t)\cos(2\pi x))/1000, \quad (x,t) \in Q_T$
and the input data are as follows:

$f(x,t) = (1/1000)(-1 + 3e^t + 30\pi \cos^2(3\pi t) + 4\pi^2)\cos(2\pi x)\cos(t), (x,t) \in Q_T$

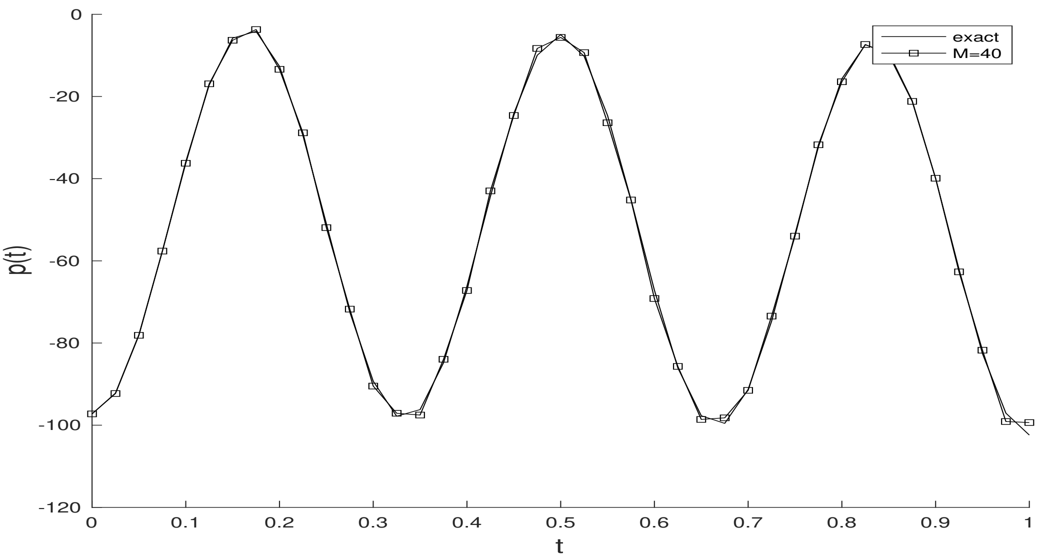
$$\varphi(x) = (\cos(2\pi x))/1000, \quad \psi(x) = 0, \quad h(t) = -\cos(t)/1000. \quad x \in [0,1], \quad t \in [0,1]$$

$$p(t) = -3e^t - 30\pi\cos^2(3\pi t), \quad t \in [0,1]$$

The initial guess was$p_0 = -3 - 30\pi$. It is easy to verify the input data for the conditions of the Theorems 1-3. The numerical investigation begins with the ideal situation when no noise included, i.e., $q = 0$ in (26). The objective function (23) represented Figure 8(a), and a speed declining convergence is seen for achieving a shorter order tolerance $O(10^{-15})$ in below 50 iterations. Figure 8(b) shows numerical results for $p(t)$ and accurate appears at is observed.



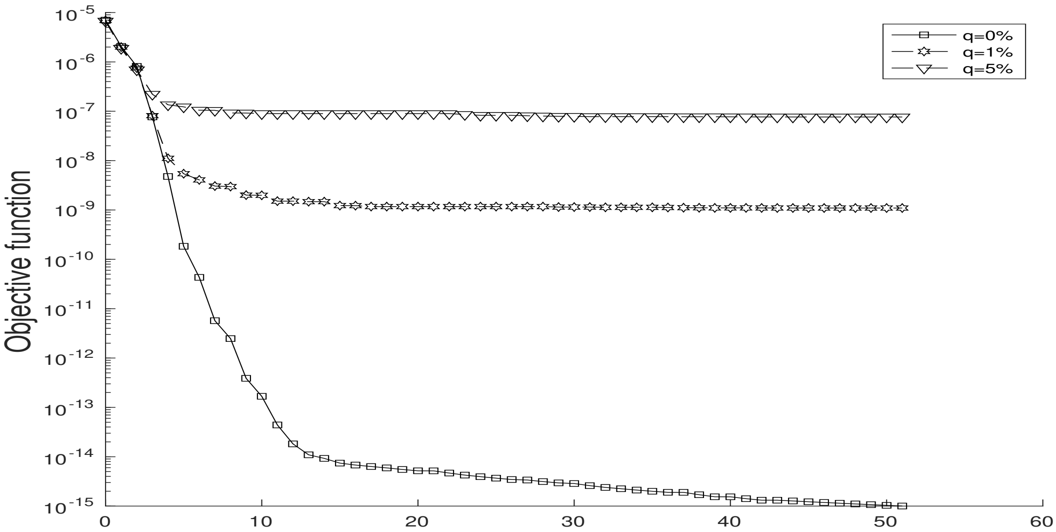
(a)



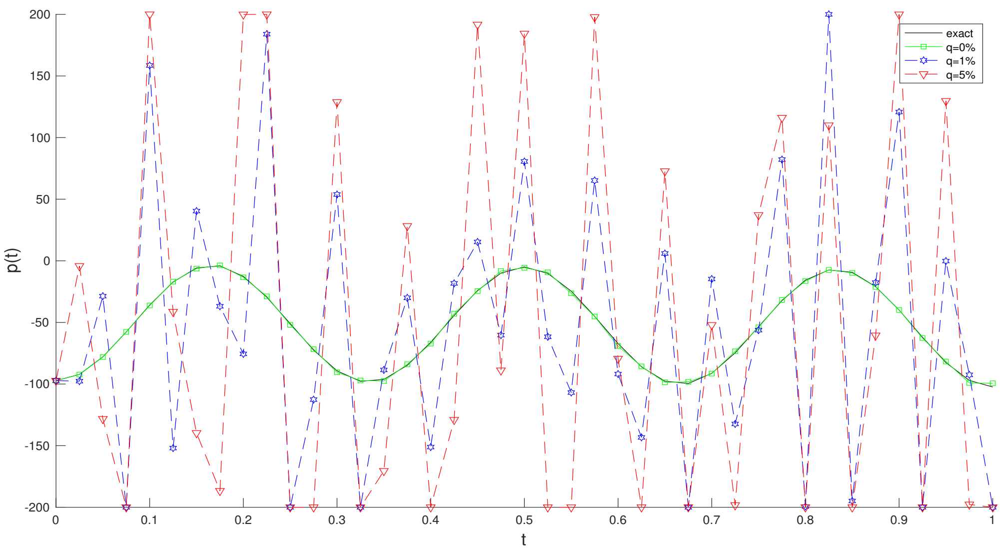
(b)

Figure 8: (a) objective function (23) and (b) $p(t)$ with noise free and without regularization.

In this case, we perturb the measured data with $q \in \{1, 5\}$%noise added as in equation (26). In the absence of regularization, the associated numerical results are presented in Figure 9. From Figure 9(a) the criterion yields the iteration number =50, revealing that the objective function minimization (23) has converged to small stationary values of orders $O(10^{-7})$ and $O(10^{-9})$) for $q \in \{1, 5\}$%noise respectively. As seen in Figure 9(b), the numerical solution of p(t) diverges from the exact solution but remains on the same path when the value of additive noise increases in equation (26).



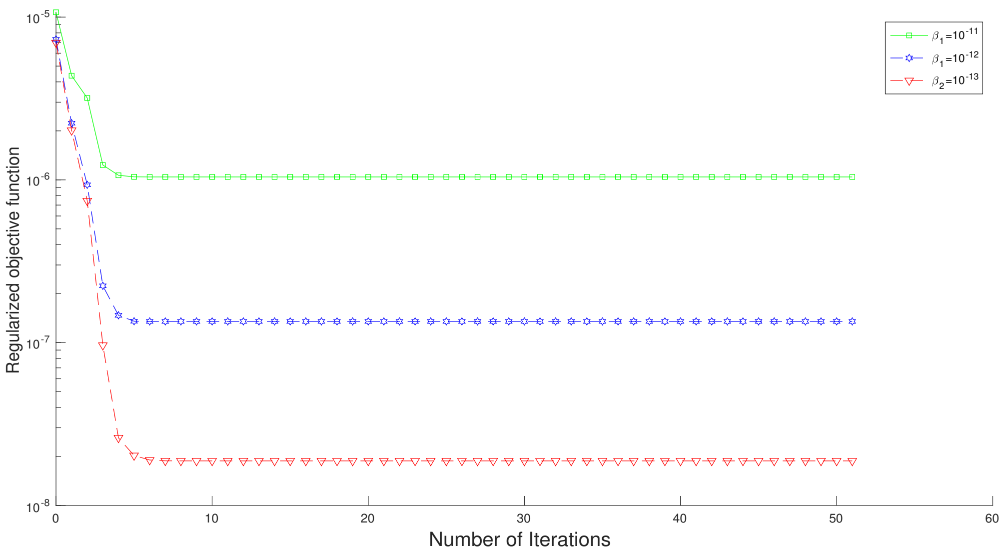
(a)



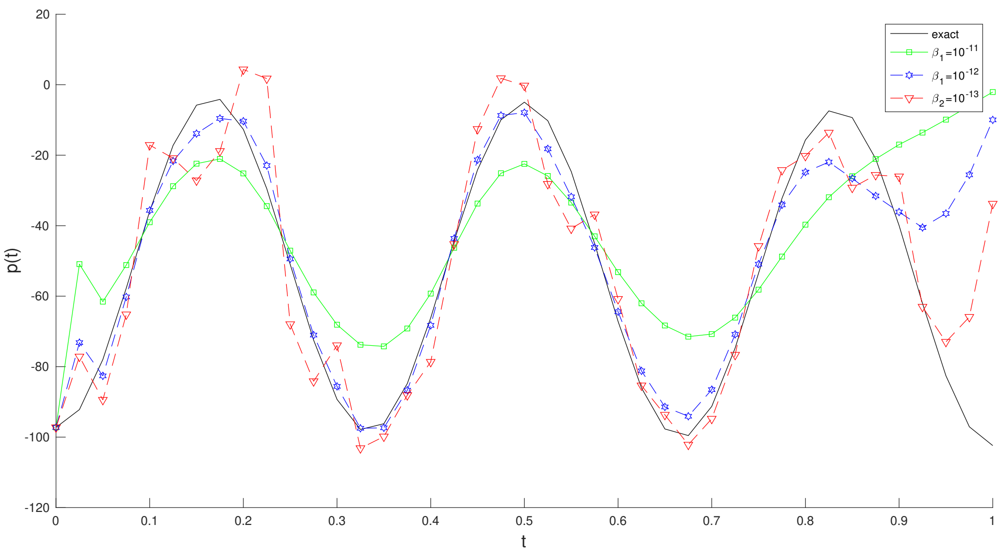
(b)

Figure 9: (a) objective function (23) and (b) Reconstruction of $p(t)$, for different noise level $q \in 1, 5$ %and no regularization.

To restore the stability some regularization should be applied. To replicate real input data, noise of $q \in \{1, 5\}$%is included with regularization$\beta \in 0, 10^{-13}, 10^{-12}, 10^{-11}$. Figure 10(a) and Figure 11(a), the criterion yields the iteration number also 50, revealing that the objective function minimization (23). Figure 10(b) and Figure 11(b), show the potential unknown coefficient $p(t)$. These figures show that results are almost completely smooth, especially in the range $[0, 0.9]$, before instabilities begin to show up when noise levels increase from 1%to 5%. A very excellent agreement is established when $\beta = 10{-}12$ is selected.

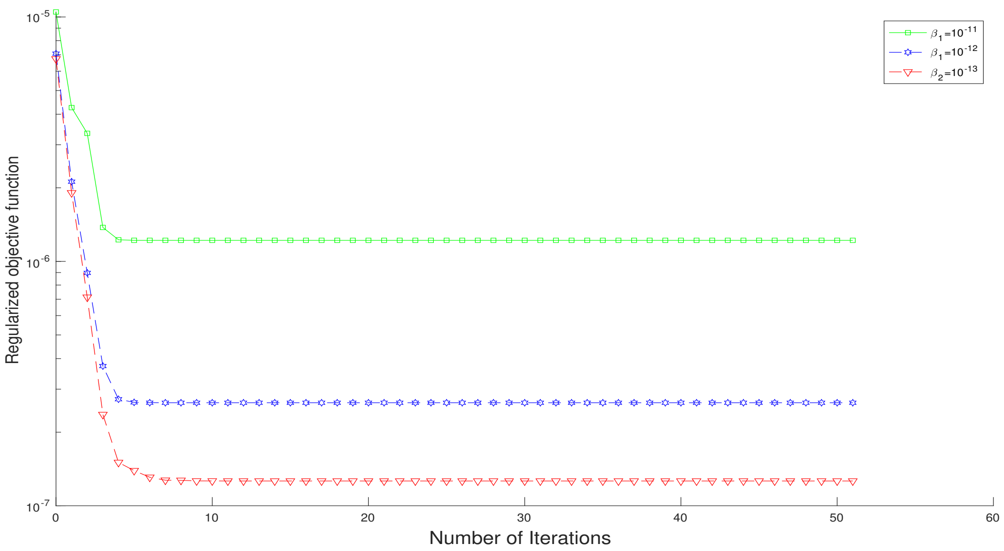


(a)

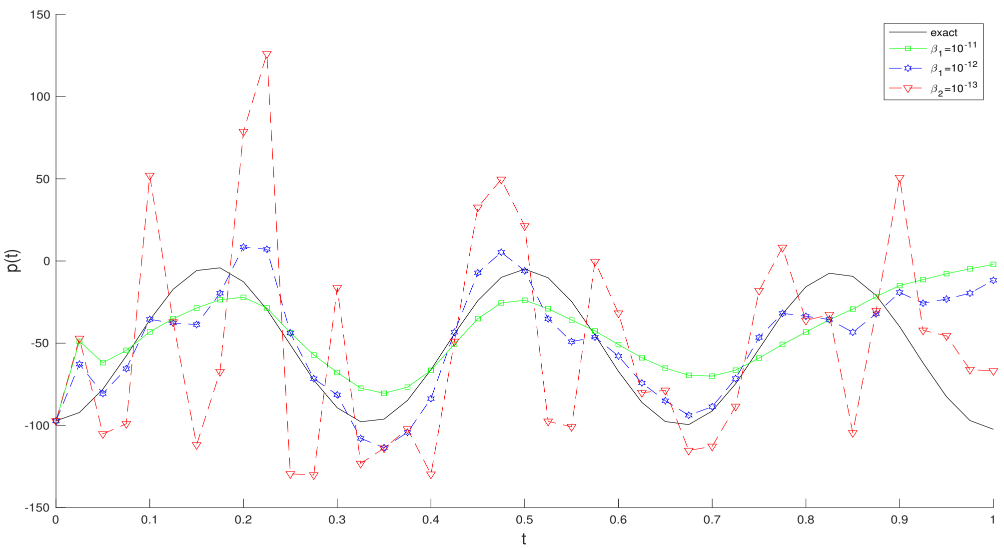


(b)

Figure 10: (a) objective function (23) and (b) $p(t)$, for $q = 1\%$noise and $\beta = \{10^{-13}, 10^{-12}, 10^{-11}\}$.



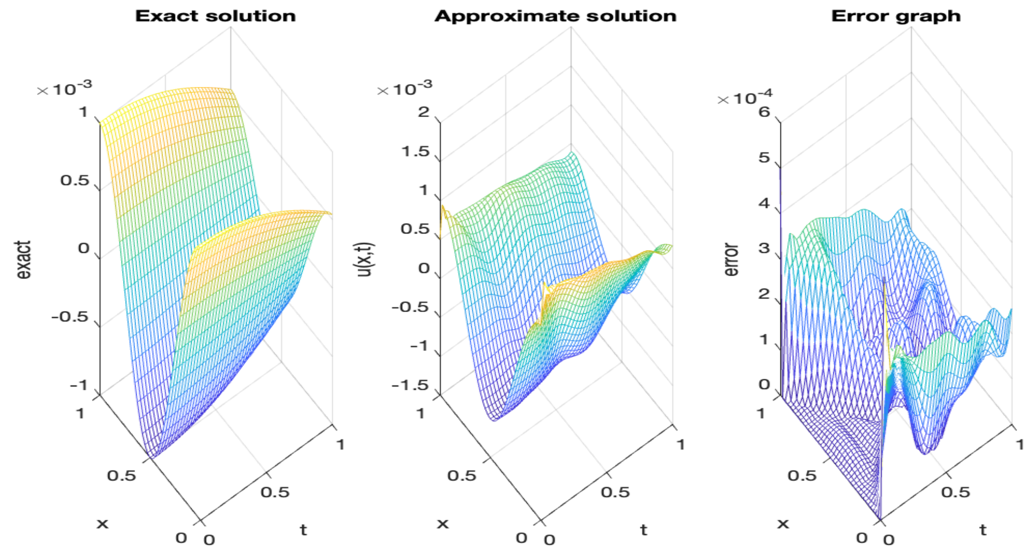
(a)



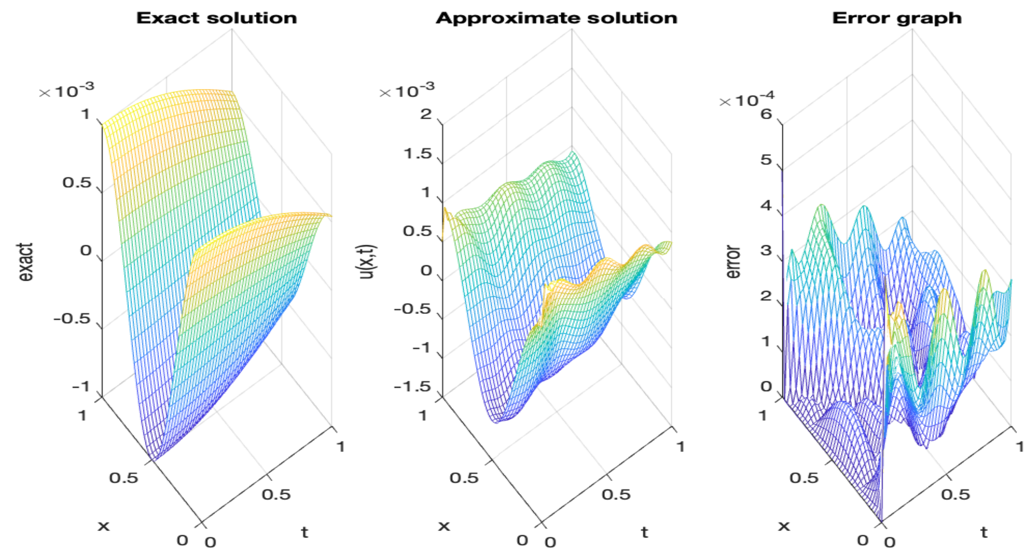
(b)

Figure 11: (a) objective function (23) and (b) $p(t)$, for $q = 5\%$noise and $\beta = \{10^{-13}, 10^{-12}, 10^{-11}\}$.

The numerical and exact temperatures $u(x,t)$, with $q = 1\%$noise, $\beta = 10^{-12}$, $q = 5\%$noise, $\beta = 10^{-12}$, as well as the absolute error between them are illustrated in Figures 12.



(a)



(b)

Figure 12: both accurate and numerical $u(x,t)$, with $q = 1$ %and $q = 5$%noise and $\beta = 10^{-12}$, as well as the absolute error between them.

6. Conclusions

The second order hyperbolic inverse problem to identify numerically the potential coefficient has been investigated under initial and nonlocal boundary conditions and overdetermination data. The finite difference scheme, in cooperation with the trapezoidal rule has been used for direct problem. Therefore, to reconstruct the stability, Tikhonov's regularization was employed. Stable results are obtained under various noise levels.

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