Advances in the Theory of Nonlinear Analysis and its Applications 7 (2023) No. 4, 181-191.
https://doi.org/10.17762/atnaa.v7.i4.315
Available online at www.https://atnaea.org/
Research Article

Advances in the Theory of Nonlinear Analysis and its Applications
ISSN: 2587-2648
Peer-Reviewed Scientific Journal

An Application of generalized Fibonacci-like polynomial on New Subfamilies of Regular and Bi-univalent Functions

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Abstract

The present paper introduces the novel subfamilies of regular and bi-univalent functions Σ. The prominent group of Fibonacci polynomial $\tilde{\mathrm{V}}_n(x, \mathrm{y})$ is utilized with subordination between regular functions in order to shape these subfamilies. In addition we derive coefficients inequalities for functions belonging to these subfamilies. Various results are exposed as separate cases of the current conclusions.

Keywords: (Regular functions; Bi-univalent functions; Generalized Bivariate FibonacciLike polynomials; Horadam polynomial; Chebyshev polynomials; subordination.).
2010 MSC: 49J20, 49K20, 35Q35, 93C20.

1. Introduction and Fundamental Notions

Fibonacci polynomials, Lucas polynomials, Chebychev polynomials, Pell polynomials, Horadam polynomial and their generalizations are widespread in the applied sciences such as Engineering, Physics and Numerical analysis (see [1]). Many polynomials are used in the geometric functions theory; they have a great impact on researchers of mathematics. Some fundamental concepts in geometric function theory, the

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Received September 22, 2023; Accepted: November 30, 2023; Online: December 23, 2023.

well-known subordination notion and certain renowned problems like initial coefficients estimate problem are taken into consideration in these new subfamilies of regular functions. According to the special relationship between polynomials and regular functions, this helps us introduce two new families of regular and bi-univalent functions in this work. After that, we have obtained the upper bounds for the initial coefficients of functions that belongs to these new subfamilies. In addition, we present several corollaries and remarks at the end of the main results.
The prominent Fibonacci polynomial is defined as follows

The Fibonacci polynomials are a polynomial sequence which can be considered as a generalization of the Fibonacci numbers by a recurrence relation:

$$\mathcal{F}_n = \mathcal{F}_{n-1} + \mathcal{F}_{n-2}, \mathcal{F}_0 = 0, \mathcal{F}_1 = 1, (\mathfrak{n} \geq 2).$$

The researchers introduced a new generalization of the Fibonacci polynomial that is called generalized bivariate Fibonacci-like polynomial in [2].
For $\mathfrak{n} \geq 2$, the generalized bivariate Fibonacci-like polynomials are defined by the recurrence relation

$$\tilde{V}_{\mathrm{n}}(x, \mathrm{y}) = \mathrm{p}x\tilde{V}_{\mathrm{n}-1}(x, \mathrm{y}) + \mathrm{qy}\tilde{\mathrm{V}}_{\mathrm{n}-2}(x, \mathrm{y}),$$

where p, q positive integers, $\mathrm{x}, \mathrm{y} \in \mathbb{R}, \tilde{\mathrm{V}}_0(\mathrm{x}, \mathrm{y}) = \mathrm{a}, \tilde{\mathrm{V}}_1(\mathrm{x}, \mathrm{y}) = \mathrm{b}, \mathrm{px}, \mathrm{qy} \neq 0$ and

$$(px)^2 + 4q\mathrm{y} \neq 0$$

The Fibonacci-like polynomials have generating function see [2]

$$\tilde{\mathrm{V}}^{(x,\mathrm{y})}(\mathrm{v}) = \sum \tilde{\mathrm{V}}_n(x, \mathrm{y})\mathrm{v}^n = \frac{a + (b - apx)\mathrm{v}}{a - pxz - q\mathrm{y}\mathrm{v}^2}.$$

For the special cases of p, q, a, b and y we can get the polynomials given in the following Table

(x, y)	(a, b)	(p, q)	$\tilde{\mathrm{V}}_n(x, \mathrm{y})$
(x, y)	$(0, 1)$	$(1, 1)$	Bivariate Fibonacci $F_n(x, \mathrm{y})$
$(x, 1)$	$(0, 1)$	$(1, 1)$	Fibonacci $F_n(x)$
$(x, 1)$	$(0, 1)$	$(2, 1)$	Pell, $P_n(x)$
(x, y)	$(2, x)$	$(1, 1)$	Bivariate Lucas $L_n(x, \mathrm{y})$
$(t, -1)$	$(1, 2t)$	$(2, 1)$	Chebyshev of the second kind $U_n(x)$
$(x, 1)$	(a, bx)	(p, q)	Horadam $H_{n+1}(x)$

Let $\hbar$ be function of the form

$$\hbar(\mathrm{v}) = \mathrm{v} + \sum_{m=2}^{\infty} \varepsilon_k \mathrm{v}^m. \tag{1}$$

which belongs to $\hat{A}$ where $\hat{A}$ is the class of regular functions defined on the disk

$$\mathrm{U} = \{\mathrm{v} \in \mathbb{C} : |\mathrm{v}| < 1\},$$

with $\hbar(0) = \hbar'(0) - 1 = 0$ and let S be the subclass of $\hat{\mathrm{A}}$ consisting of the form (1) which are also univalent in U. The Koebe's Covering Theorem [3] states for each $\hbar \in S$ the image $t = \hbar(\mathrm{v}), \mathrm{v} \in \mathrm{U}$, in the t-plane contains the disk$\{t : |t| < 1/4\}$. From this theorem, every function $\hbar \in S$ has an inverse $\hbar^{-1}$ which holds

$$\hbar^{-1}(\hbar(v)) = \mathrm{v}, (\mathrm{v} \in \mathrm{U})$$

and,

$$\hbar\left(\hbar^{-1}(t)\right)=t,\left(|t|<r_0(\hbar),r_0(\hbar)\geq\frac{1}{4}\right),$$

where

$$f(t)=\hbar^{-1}(t)=t-\varepsilon_2t^2+\left(2\varepsilon_2^2-\varepsilon_3\right)t^3 \\ -\left(5\varepsilon_2^3-5\varepsilon_2\varepsilon_3+\varepsilon_4\right)t^4+\cdots \tag{2}$$

A function $\hbar\in\mathbb{A}$ is called bi-univalent in U if both $\hbar$ and $\hbar^{-1}$ are univalent in U. The class of all bi-univalent functions in U denoted by Σ. This family was introduced by Lewin [4] and showed that $|\varepsilon_2|\leq 1.5$ for the function in the family Σ. Recently, Brannan and Clunie [5] conjectured that $|\varepsilon_2|\leq\sqrt{2}$. Netanyahu in [6] proved that $|\varepsilon_2|=\frac{4}{3}$. Recently, many authors introduced and investigated several interesting subclasses of bi-univalent functions [7, 8, 9, 10, 11].
A regular function $\hbar$ is subordinate to regular function f, written

$$\hbar\prec\mathrm{f}\text{ or }\hbar(\mathrm{v})\prec\mathrm{f}(\mathrm{v}),(\mathrm{v}\in\mathrm{U}). \tag{3}$$

If there is regular function $\mathrm{k}:\mathrm{U}\to\mathrm{U}$, with $\mathrm{k}(0)=0$ and $|\mathrm{k}(\mathrm{v})|<1$ such that

$$\hbar(\mathrm{v})=\mathrm{f}(\mathrm{k}(\mathrm{v})),\mathrm{v}\in\mathrm{U}.$$

It follows from the definition that

$$\hbar(0)=\mathrm{f}(0)\text{ and }\hbar(\mathrm{U})\subset\mathrm{f}(\mathrm{U}).$$

For more details about the concepts of subordination (see [[12], [13], and [11]]).

In this present work, we introduce a new subclass of bi-univalent functions satisfying Subordinate conditions and defined by Fibonacci polynomial. Also, we obtain the coefficient estimates for $|\varepsilon_2|$ and $|\varepsilon_3|$ for functions of the new class.

2. The Family $\mathcal{M}^{p,q,x,y}(\varrho,\xi,\Sigma;\tilde{V})$ and the coefficients Bounds

Definition 2.1. *We say that, $\hbar\in\Sigma$ given by (1) belong to $\mathcal{M}^{p,q,x,y}(\varrho,\xi,\Sigma;\tilde{\mathrm{V}})$ if it satisfies the following conditions:*

$$(1-\xi)\left(\frac{v\hbar'(v)}{\hbar(v)}+\frac{v\hbar''(v)}{\hbar'(v)}\right)+\xi\left(\frac{\varrho v^2\hbar''(v)+\hbar'(v)}{\varrho v\hbar'(v)+(1-\varrho)\hbar(v)}\right) \\ \prec\mathfrak{X}(v)=\tilde{V}^{(x,\mathrm{y})}(v)+1-a,(v\in U)$$

and

$$(1-\xi)\left(\frac{t\mathrm{k}'(t)}{\mathrm{k}(t)}+\frac{t\mathrm{k}''(t)}{\mathrm{k}'(t)}\right)+\xi\left(\frac{\varrho t^2\mathrm{k}''(t)+t\mathrm{k}'(t)}{\varrho t\mathrm{k}'(t)+(1-\varrho)\mathrm{k}(t)}\right) \\ \prec\mathfrak{X}(t)=\tilde{\mathrm{V}}^{(x,y)}(t)+1-a,(t\in\mathrm{U}),$$

where $0\leq\varrho\leq 1, 0\leq\xi\leq 1$ and the function $\mathrm{k}=\hbar^{-1}$ is given by (2).

It is interesting to note that the special values of ϱ,ξ and $\tilde{V}$ lead the family $\mathcal{M}^{p,q,x,y}(\varrho,\xi,\Sigma;\tilde{\mathrm{V}})$ to various subfamily, we illustrate the following subfamily:

Remark 2.2. *For $\varrho=0,1$, in $\mathcal{M}^{p,q,x,y}(\varrho,\xi,\Sigma;\tilde{V})$, we get the subfamilies $\mathcal{M}^{p,q,x,y}(0,\xi,\Sigma;\tilde{\mathrm{V}})$ and $\mathcal{M}^{p,q,x,y}(1,\xi,\Sigma;\tilde{\mathrm{V}})$, respectively.*

Remark 2.3. *Let* $\tilde{\mathrm{V}}^{(x,\mathrm{y})}$ *is Horadam polynomial with* $\mathrm{b} = \mathrm{bx}$ *and* $\mathrm{y} = 1$*, we have* $\mathcal{M}^{p,q,x,1}(\varrho, \xi, \Sigma; \tilde{V})$*.In this case for* $\varrho = 0, 1$ *be in this family. Then we get new families* $\mathcal{M}^{p,q,x,1}(0, \xi, \Sigma; \tilde{\mathrm{V}})$ *and* $\mathcal{M}^{p,q,x,1}(1, \xi, \Sigma; \tilde{\mathrm{V}})$ *respectively.*

Remark 2.4. Let $\tilde{V}^{(x,\mathrm{y})}$ *is Chebyshev polynomials with* $\mathrm{p} = 2$*,* $\mathrm{b} = 2\mathrm{t}, \mathrm{x} = \mathrm{t}, \mathrm{y} = -1, \mathrm{q} = 1, \mathrm{a} = 1$ *in Theorem 1 we have* $\mathcal{M}^{2,1,x,-1}(\varrho, \xi, \Sigma; \tilde{V})$*. In this case for* $\varrho = 0, 1$ *be in this family. Then we get new families* $\mathcal{M}^{2,1,x,-1}(0, \xi, \Sigma; \tilde{V})$ *and* $\mathcal{M}^{2,1,x,-1}(1, \xi, \Sigma; \tilde{V})$ *respectively.*

Remark 2.5. *Let* $\tilde{V}^{(x,\mathrm{y})}$ *is Fibonacci polynomials with* $\mathrm{p} = \mathrm{q} = 1$*,* $\mathrm{b} = 1, \mathrm{y} = 1, \mathrm{a} = 0$*, we have* $\mathcal{M}^{1,1,x,1}(\varrho, \xi, \Sigma; \tilde{V})$*. In this case for* $\varrho = 0, 1$ *be in this family. Then we get new families* $\mathcal{M}^{1,1,x,1}(0, \xi, \Sigma; \tilde{V})$ *and* $\mathcal{M}^{1,1,x,1}(1, \xi, \Sigma; \tilde{V})$ *respectively.*

Remark 2.6. *Putting* $\xi = 1$ *in the family* $\mathcal{M}^{p,q,x,\mathrm{y}}(\varrho, \xi, \Sigma; \tilde{V})$*, we have* $\mathcal{H}^{p,q,x,\mathrm{y}}_{n,\Sigma,\gamma}(h(\mathrm{v}))$ *in this case for* $\varrho = 0, 1$ *we have* $\mathcal{H}^{p,q,x,\mathrm{y}}_{n,\Sigma,0}(h(\mathrm{v}))$ *and* $\mathcal{H}^{p,q,x,\mathrm{y}}_{n,\Sigma,1}(h(\mathrm{v}))$*, respectively [14].*

In this section, we investigate coefficient estimates for the function in the family $\mathcal{M}^{p,q,x,\mathrm{y}}(\varrho, \xi, \Sigma; \tilde{V})$

Theorem 2.7. *For,* $0 \le \xi \le 1$ *and* $0 \le \varrho \le 1$*, let* $\hbar \in \Sigma$ *given by (1) belong to* $\mathcal{M}^{p,q,x,\mathrm{y}}(\varrho, \xi, \Sigma; \tilde{V})$*, then*

$$|\varepsilon_2| \le \frac{|b|\sqrt{|b|}}{\sqrt{|[(8(1-\xi)+2\xi(1+2\varrho)) - (5(1-\xi)+\xi(1+\varrho)^2)]\, b^2 - (pbx+aq\mathrm{y})(3(1-\xi)+\xi(1+\varrho))^2|}} \tag{4}$$

$$|\varepsilon_3| \le \frac{b^2}{(3(1-\xi)+\xi(1+\varrho))^2} + \frac{|b|}{(8(1-\xi)+2\xi(1+2\varrho))} \tag{5}$$

Proof. Since $\hbar \in \mathcal{M}^{p,q,x,\mathrm{y}}(\varrho, \xi, \Sigma; \tilde{V})$, there exist regular functions $r, \theta : \mathrm{U} \to \mathrm{U}$, such that

$$\mathbf{r}(\mathrm{v}) = \sum_{1}^{\infty} \mathbf{r}_k \mathrm{v}^k, \boldsymbol{\Theta}(t) = \sum_{1}^{\infty} \boldsymbol{\Theta}_k t^k \quad \text{and} \quad |\mathbf{r}_k| \le 1, |\boldsymbol{\Theta}_k| \le 1, \tag{6}$$

By Definition 2.1

$$(1-\xi)\left(\frac{\mathrm{v}\hbar'(\mathrm{v})}{\hbar(\mathrm{v})} + \frac{\mathrm{v}''(\mathrm{v})}{\hbar'(\mathrm{v})}\right) + \xi\left(\frac{\varrho \mathrm{v}^2\hbar''(\mathrm{v}) + \mathrm{v}\hbar'(\mathrm{v})}{\varrho \mathrm{v}\hbar'(\mathrm{v}) + (1-\varrho)\hbar(\mathrm{v})}\right) = \mathfrak{X}(\mathrm{f}(\mathrm{v})) \tag{7}$$

$$(1-\xi)\left(\frac{t\mathrm{k}'(t)}{\mathrm{k}(t)} + \frac{t\mathrm{k}''(t)}{\mathrm{k}'(t)}\right) + \xi\left(\frac{\varrho t^2\mathrm{k}''(t) + t\mathrm{k}'(t)}{\varrho t\mathrm{k}'(t) + (1-\varrho)\mathrm{k}(t)}\right) = \mathfrak{X}(\Theta(t)) \tag{8}$$

It is possible to write that

$$\mathfrak{X}(\mathrm{r}(\mathrm{v})) = 1 + \tilde{V}_1(x,\mathrm{y})\mathrm{r}(\mathrm{v}) + \tilde{V}_2(x,\mathrm{y})(\mathrm{r}(\mathrm{v}))^2 + \tilde{V}_3(x,\mathrm{y})(\mathrm{r}(\mathrm{v}))^3 + \cdots \tag{9}$$

$$\mathfrak{X}(\theta(t)) = 1 + \tilde{V}_1(\mathrm{x},\mathrm{y})\Theta(t) + \tilde{\mathrm{V}}_2(\mathrm{x},\mathrm{y})(\theta(t))^2 + \tilde{\mathrm{V}}_3(\mathrm{x},\mathrm{y})(\theta(t))^3 + \cdots \tag{10}$$

Put (9) and (10) into (7) and (8) respectively and compare the coefficients, thus we get

$$(3(1-\xi)+\xi(1+\varrho))\varepsilon_2 = \tilde{V}_1(x,\mathrm{y})\mathbf{r}_1 \tag{11}$$

$$(8(1-\xi)+2\xi(1+2\varrho))\varepsilon_3 - \left(5(1-\xi)+\xi(1+\varrho)^2\right)\varepsilon_2^2 = \tilde{V}_1(x,\mathrm{y})\mathbf{r}_2 + \tilde{V}_2(x,\mathrm{y}){\mathbf{r}_1}^2 \tag{12}$$

$$-(3(1-\xi)+\xi(1+\varrho))\varepsilon_2 = \tilde{V}_1(x,\mathrm{y})\boldsymbol{\theta}_1 \tag{13}$$

$$
\begin{aligned}
&-(8(1-\xi)+2\xi(1+2\varrho))\varepsilon_3+\{2(8(1-\xi)\\
&+2\xi(1+2\varrho))-\left(5(1-\xi)+\xi(1+\varrho)^2\right)\}\,{\varepsilon_2}^2\\
&=\check{\mathrm{V}}_1(\mathrm{x},\mathrm{y})\boldsymbol{\theta}_2+\check{\mathrm{V}}_2(\mathrm{x},\mathrm{y}){\boldsymbol{\theta}_1}^2
\end{aligned}
\tag{14}
$$

From the equations (11) and (13), we can write

$$
\mathfrak{r}_1=-\boldsymbol{\theta}_1 \tag{15}
$$

$$
\begin{aligned}
2(3(1-\xi)+\xi(1+\varrho))^2{\varepsilon_2}^2&=\tilde{V}_1{}^2(x,\mathrm{y})\left({\mathbf{r}_1}^2+{\Theta_1}^2\right)\\
{\varepsilon_2}^2&=\frac{\tilde{\mathrm{V}}_1{}^2(x,\mathrm{y})\left({\mathrm{r}_1}^2+{\Theta_1}^2\right)}{2(3(1-\xi)+\xi(1+\varrho))^2}
\end{aligned}
\tag{16}
$$

By summing the equation (12) and (14), we reduce it

$$
\begin{aligned}
&\left\{\left[2(8(1-\xi)+2\xi(1+2\varrho))-2\left(5(1-\xi)+\xi(1+\varrho)^2\right)\right]\right.\\
&\left.+2(3(1-\xi)+\xi(1+\varrho))^2\right\}{\varepsilon_2}^2\\
&=\tilde{V}_1(x,\mathrm{y})\left(\mathbf{r}_2+\Theta_2\right)+\tilde{V}_2(x,\mathrm{y})\left({\mathbf{r}_1}^2+{\Theta_1}^2\right)
\end{aligned}
\tag{17}
$$

Put (15), (16) in (17), we get

$$
\begin{aligned}
&\varepsilon_2^2=\frac{\tilde{\mathrm{V}}_1^3(\mathrm{x},\mathrm{y})\left(\mathrm{r}_2+\boldsymbol{\theta}_2\right)}{[2(8(1-\xi)+2\xi(1+2\varrho))-(10(1-\xi)+2\xi(1+\varrho)^2)]}\\
&\tilde{\mathrm{V}}_1^2(\mathrm{x},\mathrm{y})-\tilde{V}_2(x,\mathrm{y})2(3(1-\xi)+\xi(1+\varrho))^2
\end{aligned}
$$

Using (6) in above relation we get (4)
By subtracting (14) from (12), we have

$$
2(8(1-\xi)+2\xi(1+2\varrho))\left\{\varepsilon_3-\varepsilon_2^2\right\}=\tilde{V}_1(x,\mathrm{y})\left(\mathrm{r}_2-\theta_2\right)
$$

Using inequality (6) and relation (16) in above relation we get (5)

$$
|\varepsilon_3|\le\frac{\mathrm{b}^2}{(3(1-\xi)+\xi(1+\varrho))^2}+\frac{\left|\tilde{V}_1(x,y)\right|}{(8(1-\xi)+2\xi(1+2\varrho))}
$$

By giving different values to the parameters in Theorem 1, we obtain some bounds on the coefficients ε_2 and ε_3 of bi-univalent functions □

Corollary 2.8. *(i) Let $\hbar$ given by (1) be in the subfamily $\mathcal{M}^{p,q,x,\mathrm{y}}(0,\xi,\Sigma;\tilde{V})$. Then*

$$
\begin{aligned}
|\varepsilon_2|&\le\frac{|b|\sqrt{|b|}}{\sqrt{|b^2(3-2\xi)-(pbx+aqy)(3-2\xi)^2|}}\\
|\varepsilon_3|&\le\frac{\mathrm{b}^2}{(3-2\xi)^2}+\frac{|\mathrm{F}_1(x,\mathrm{y})|}{(8-6\xi)}
\end{aligned}
$$

(ii) Let $\hbar$ given by (1) be in the subfamily $\mathcal{M}^{p,q,x,\mathrm{y}}(1,\xi,\Sigma;\tilde{V})$. Then

$$
\begin{aligned}
|\varepsilon_2|&\le\frac{|b|\sqrt{|b|}}{\sqrt{|b^2(3-\xi)-(pbx+aqy)(3-2\xi)^2|}}\\
|\varepsilon_3|&\le\frac{\mathrm{b}^2}{(3-2\xi)^2}+\frac{|\mathrm{b}|}{(8-2\xi)}
\end{aligned}
$$

Corollary 2.9. *(i) Let $\hbar$ given by (1) be in the subfamily $\mathcal{M}^{p,q,x,1}(\varrho,\xi,\Sigma;\tilde{V})$. Then*

$$|\varepsilon_2| \le \frac{|bx|\sqrt{|bx|}}{\sqrt{\left|\begin{array}{c}\left[(8(1-\xi)+2\xi(1+2\varrho))-\left(5(1-\xi)+\xi(1+\varrho)^2\right)\right](bx)^2\\ -\left(pbx^2+aq\right)(3(1-\xi)+\xi(1+\varrho))^2\end{array}\right|}}$$

$$|\varepsilon_3| \le \frac{(bx)^2}{(3(1-\xi)+\xi(1+\varrho))^2}+\frac{|bx|}{(8(1-\xi)+2\xi(1+2\varrho))}$$

(ii) Let $\hbar$ given by (1) be in the subfamily $\mathcal{M}^{p,q,x,1}(0,\xi,\Sigma;\tilde{V})$. Then

$$|\varepsilon_2| \le \frac{|bx|\sqrt{|bx|}}{\sqrt{|[(3-2\xi)](bx)^2-\left(pbx^2+aq\right)(3-\xi)^2|}}$$

$$|\varepsilon_3| \le \frac{(\mathrm{bx})^2}{(3-2\xi)^2}+\frac{|\mathrm{bx}|}{(8-6\xi)}.$$

(iii) Let $\hbar$ given by (1) be in the subfamily $\mathcal{M}^{p,q,x,1}(0,\xi,\Sigma;\tilde{V})$. Then

$$|\varepsilon_2| \le \frac{|\mathrm{bx}|\sqrt{|\mathrm{bx}|}}{\sqrt{|[(3-\xi)](\mathrm{bx})^2-(\mathrm{pbx}+\mathrm{aq})(3-\xi)^2|}}$$

$$|\varepsilon_3| \le \frac{(\mathrm{bx})^2}{(3-\xi)^2}+\frac{|\mathrm{bx}|}{(8-2\xi)}.$$

Corollary 2.10. *(i) Let $\hbar\in\Sigma$ given by (1) in the subfamily $\mathcal{M}^{2,1,x,-1}(\varrho,\xi,\Sigma;\tilde{V})$ and $\tilde{\mathrm{V}}^{(x,\mathrm{y})}$ be Chebyshev polynomials with $\mathrm{p}=2,\ \mathrm{b}=2\mathrm{t},\mathrm{x}=\mathrm{t},\mathrm{y}=-1,\mathrm{q}=1,\mathrm{a}=1$. Then*

$$|\varepsilon_2| \le \frac{|2t|\sqrt{|2t|}}{\sqrt{|[(8(1-\xi)+2\xi(1+2\varrho))-(5(1-\xi)+\xi(1+\varrho)^2)]\,4t^2-\left(4t^2-1\right)(3(1-\xi)+\xi(1+\varrho))^2|}}$$

$$|\varepsilon_3| \le \frac{4t^2}{(3(1-\xi)+\xi(1+\varrho))^2}+\frac{|2\mathrm{t}|}{(8(1-\xi)+2\xi(1+2\varrho))}$$

(ii) Let $\hbar\in\Sigma$ given by (1) in the subfamily $\mathcal{M}^{2,1,x,-1}(0,\xi,\Sigma;\tilde{V})$. Then

$$|\varepsilon_2| \le \frac{|2t|\sqrt{|2t|}}{\sqrt{|[3-2\xi]4t^2-\left(4t^2-1\right)(3-2\xi)^2|}}$$

$$|\varepsilon_3| \le \frac{4\mathrm{t}^2}{(3-2\xi)^2}+\frac{|\mathrm{t}|}{4-3\xi}$$

(iii) Let $\hbar\in\Sigma$ given by (1) in the subfamily $\mathcal{M}^{2,1,x,-1}(1,\xi,\Sigma;\tilde{V})$. Then

$$|\varepsilon_2| \le \frac{|2\mathrm{t}|\sqrt{|2\mathrm{t}|}}{\sqrt{|[3-\xi]4\mathrm{t}^2-\left(4\mathrm{t}^2-1\right)(3-\xi)^2|}}$$

$$|\varepsilon_3| \le \frac{4\mathrm{t}^2}{(3-\xi)^2}+\frac{|t|}{4-\xi}$$

Corollary 2.11. *(i) Let $\hbar\in\Sigma$ given by (1) in the subfamily $\mathcal{M}^{1,1,x,1}(\varrho,\xi,\Sigma;\ \tilde{V})$ and $\tilde{\mathrm{V}}^{(x,\mathrm{y})}$ be Fibonacci polynomials with $\mathrm{p}=\mathrm{q}=1,\ \mathrm{b}=1,\mathrm{y}=1,\mathrm{a}=0$. Then*

$$|\varepsilon_2| \le \frac{1}{\sqrt{|[(8(1-\xi)+2\xi(1+2\varrho))-(5(1-\xi)+\xi(1+\varrho)^2)]-(3(1-\xi)+\xi(1+\varrho))^2x|}}$$

$$|\varepsilon_3| \le \frac{1}{(3(1-\xi)+\xi(1+\varrho))^2}+\frac{1}{(8(1-\xi)+2\xi(1+2\varrho))}$$

(ii) Let $\hbar \in \Sigma$ given by (1) in the subfamily $\mathcal{M}^{1,1,x,1}(0,\xi,\Sigma;\ddot{V})$. Then

$$|\varepsilon_2| \leq \frac{1}{\sqrt{|[3-2\xi]-(3-2\xi)^2 x|}}$$
$$|\varepsilon_3| \leq \frac{1}{(3-2\xi)^2} + \frac{1}{(8-6\xi)}$$

(iii) Let $\hbar \in \Sigma$ given by (1.1) in the class $\mathcal{M}^{1,1,x,1}(1,\xi,\Sigma;\tilde{V})$

$$|\varepsilon_2| \leq \frac{1}{\sqrt{|3-\xi-(3-\xi)^2 x|}}$$
$$|\varepsilon_3| \leq \frac{1}{(3-\xi)^2} + \frac{1}{8-2\xi}$$

3. The Family $\mathfrak{B}^{(p,q,x,y)}(\gamma,d;\tilde{V})$ and the coefficients Bounds

Definition 3.1. *We say that $\hbar \in \Sigma$ given by (1) belong to $\mathfrak{B}^{(p,q,x,y)}(\gamma,d;\tilde{V})$ if it satisfies the following conditions:*

$$\frac{\mathrm{d}-\gamma}{\mathrm{d}+\gamma} + \left[\frac{2\gamma}{\mathrm{d}+\gamma}\left(1+\frac{\mathrm{v}\hbar''(\mathrm{v})}{\hbar'(\mathrm{v})}\right)\right] \prec \mathfrak{X}(\mathrm{v}) = \tilde{\mathrm{V}}^{(x,\mathrm{y})}(\mathrm{v}) + 1 - a, (\mathrm{v} \in \mathrm{U})$$

and

$$\frac{\mathrm{d}-\gamma}{\mathrm{d}+\gamma} + \left[\frac{2\gamma}{\mathrm{d}+\gamma}\left(1+\frac{t\mathrm{k}''(t)}{\mathrm{k}'(t)}\right)\right] \prec \mathfrak{X}(t) = \tilde{\mathrm{V}}^{(x,\mathrm{y})}(t) + 1 - a, \quad (t \in \mathrm{U}),$$

where , $0 < \mathrm{d} \leq 1, 0 < \mathrm{y} \leq 1$ and the function $\mathrm{k} = \hbar^{-1}$ is given by (2).

It is interesting to note that the special values of d and $\tilde{V}$ lead the family $\mathfrak{B}^{(p,q,x,y)}(\gamma,\mathrm{d};\tilde{V})$ to various subfamily, we illustrate the following subfamily

Remark 3.2. *For $d=1$, in $\mathfrak{B}^{(p,q,x,y)}(\gamma,d;\tilde{V})$, we get $\mathfrak{B}^{(p,q,x,y)}(\gamma,1;\tilde{V})$.*

Remark 3.3. *Let $\tilde{V}^{(x,y)}$ is Horadam polynomial with $b = bx$ and $y = 1$ in Theorem 2, we have $\boldsymbol{B}^{(p,q,x,1)}(\gamma,\mathrm{d};\tilde{\mathrm{V}})$.In this case for $\varrho = 0,1$ be in this family. Then we get new families $\mathfrak{B}^{(p,q,x,y)}(\gamma,0;\tilde{\mathrm{V}})$ and $\mathfrak{B}^{(p,q,x,y)}(\gamma,1;\tilde{\mathrm{V}})$ respectively.*

Remark 3.4. *Let $\tilde{V}^{(x,y)}$ is Chebyshev polynomials with $p=2, b=2t, x=t, y=-1, q=1, a=1$ in Theorem 2 we have $\mathfrak{B}^{2,1,x,-1}(\gamma,d;\tilde{V})$. In this case for $d=0,1$ be in this family. Then we get new families $\mathfrak{B}^{2,1,x,-1}(\gamma,0;\tilde{V})$ and $\mathfrak{B}^{2,1,x,-1}(\gamma,1;\tilde{V})$ respectively.*

Remark 3.5. *Let $\tilde{V}^{(x,y)}$ is Fibonacci polynomials with $\mathrm{p}=\mathrm{q}=1$, $\mathrm{b}=1, \mathrm{y}=1, \mathrm{a}=0$ in Theorem 2, we have $\mathfrak{B}^{(1,1,x,1)}(\gamma,d;\tilde{V})$. In this case for $\varrho=0,1$ be in this family. Then we get new families $\mathfrak{B}^{(1,1,x,1}(\gamma,0;\tilde{V})$ and $\mathfrak{B}^{(1,1,x,1}(\gamma,1;\tilde{V})$ respectively.*

Remark 3.6. *Putting $\mathrm{d}=\gamma$ in the family $\mathfrak{B}^{(p,q,x,y)}(\gamma,d;\tilde{V})$, we have $\mathcal{H}^{p,q,x,y}_{n,\Sigma,1}(h(v))$ [14]:*

In this section, we investigate coefficient estimates for the function family $\mathfrak{B}^{(p,q,x,y)}(\gamma,d;\tilde{V})$.

Theorem 3.7. *For, $0 < \gamma \leq 1$ and $0 \leq \mathrm{d} \leq 1$, let $\hbar \in \Sigma$ given by (1) belong to $\mathfrak{B}^{(p,q,x,y)}(\gamma,d;\tilde{V})$, then*

$$|\varepsilon_2| \leq \frac{(\mathrm{d}+\mathrm{y})|b|\sqrt{|b|}}{2\sqrt{|(\mathrm{d}+\gamma)b^2 - 4(pbx+aq\mathrm{y})|}} \tag{18}$$

$$|\varepsilon_3| \leq \frac{(\mathrm{d}+\gamma)^2 b^2}{16} + \frac{(\mathrm{d}+\gamma)\,|\mathrm{F}_1(x,\mathrm{y})|}{12\mathrm{y}} \tag{19}$$

Proof. Since $\hbar \in \mathfrak{B}^{(\boldsymbol{p},\boldsymbol{q},\boldsymbol{x},\boldsymbol{y})}(\gamma, \mathrm{d}; \tilde{\mathrm{V}})$, there exist regular functions, $\mathrm{r}, \boldsymbol{\Theta} : \mathrm{U} \to \mathrm{U}$ such that $\mathbf{r}(\mathrm{v}) = \sum_1^\infty \mathbf{r}_k \mathrm{v}^k, \Theta(t) = \sum_1^\infty \Theta_k t^k$ and $|\mathbf{r}_k| \leq 1, |\Theta_k| \leq 1$, By Definition 2.1

$$\begin{aligned}&\frac{\mathrm{d}-\gamma}{\mathrm{d}+\gamma} + \left[\frac{2\gamma}{\mathrm{d}+\gamma}\left(1+\frac{\mathrm{v}\hbar''(\mathrm{v})}{\hbar'(\mathrm{v})}\right)\right] \prec \mathfrak{X}(\mathrm{v})\\ &= \tilde{\mathrm{V}}^{(x,\mathrm{y})}(\mathrm{v}) + 1 - a, \quad (\mathrm{v} \in \mathrm{U})\end{aligned} \tag{20}$$

and

$$\begin{aligned}&\frac{\mathrm{d}-\gamma}{\mathrm{d}+\gamma} + \left[\frac{2\gamma}{\mathrm{d}+\gamma}\left(1+\frac{t\mathrm{k}''(t)}{\mathrm{k}'(t)}\right)\right] \prec \mathfrak{X}(t)\\ &= \tilde{\mathrm{V}}^{(x,\mathrm{y})}(t) + 1 - a, (t \in \mathrm{U}),\end{aligned} \tag{21}$$

Put (9) and (10) into (20) and (21) respectively and compare the coefficients, thus we get

$$\frac{4\gamma}{\mathrm{d}+\gamma}\varepsilon_2 = \tilde{\mathrm{V}}_1(x,\mathrm{y})\mathrm{r}_1 \tag{22}$$

$$\frac{12\mathrm{y}}{\mathrm{d}+\gamma}\varepsilon_3 - \frac{8\mathrm{y}}{\mathrm{d}+\gamma}{\varepsilon_2}^2 = \tilde{\mathrm{V}}_1(x,\mathrm{y})\mathrm{r}_2 + \tilde{\mathrm{V}}_2(x,\mathrm{y}){\mathrm{r}_1}^2, \tag{23}$$

$$-\frac{4\gamma}{d+\gamma}\varepsilon_2 = \tilde{V}_1(x,\mathrm{y})\theta_1, \tag{24}$$

$$\frac{16\gamma}{\mathrm{d}+\gamma}{\varepsilon_2}^2 - \frac{12\gamma}{\mathrm{d}+\gamma}\varepsilon_3 = \tilde{\mathrm{V}}_1(x,\mathrm{y})_{\Theta_2} + \tilde{\mathrm{V}}_2(x,\mathrm{y}){}_{\Theta_1}{}^2, \tag{25}$$

From the equations (22) and (24), we can write

$$\begin{aligned}\mathrm{r}_1 &= -\boldsymbol{\theta}_1\\ {\varepsilon_2}^2 &= \frac{(\mathrm{d}+\gamma)^2\,\tilde{\mathrm{V}}_{\mathbf{1}}{}^2(x,\mathrm{y})\left({\mathbf{r}_1}^2 + {\boldsymbol{\theta}_1}^2\right)}{32}\end{aligned} \tag{26}$$

By adding the equation (23) to the equation (25), we reduce it

$$\frac{8\mathrm{y}}{\mathrm{d}+\gamma}{\varepsilon_2}^2 = \tilde{\mathrm{V}}_1(x,\mathrm{y})\left(\mathrm{f}_2+\theta_2\right) + \tilde{\mathrm{V}}_2(x,\mathrm{y})\left({\mathrm{r}_1}^2 + {\Theta_1}^2\right), \tag{27}$$

Put (26) in (27), we get

$$\varepsilon_2^2 = \frac{(\mathrm{d}+\gamma)^2\tilde{\mathrm{V}}_1{}^3(x,\mathrm{y})\left(\mathbf{r}_2+\Theta_2\right)}{8(\mathrm{d}+\gamma)\tilde{\mathrm{V}}_1(x,\mathrm{y}) - 32\tilde{\mathrm{V}}_2(x,\mathrm{y})}$$

In view of (6), we conclude that (17).
By subtracting (25) from (23), we have

$$\frac{24\gamma}{\mathrm{d}+\gamma}\left(\varepsilon_3 - \varepsilon_2^2\right) = \tilde{V}_1(x,\mathrm{y})\left(\mathbf{r}_2 - \Theta_2\right)$$

Put (26) in above equation and using (6), we get

$$|\varepsilon_3| \leq \frac{(\mathrm{d}+\gamma)^2 b^2}{16} + \frac{(\mathrm{d}+\gamma)|b|}{12\gamma}$$

By giving different values to the parameters in Theorem 2, we obtain some bounds on the coefficients ε_2 and ε_3 of bi-univalent functions □

Corollary 3.8. *(i) Let $\hbar$ given by (1) be in the subfamily $\mathfrak{B}^{(\boldsymbol{p,q,x,y})}(\gamma,1;\tilde{\mathrm{V}})$. Then*

$$|\varepsilon_2| \le \frac{(1+\gamma)|\mathrm{b}|\sqrt{|\mathrm{b}|}}{2\sqrt{|(1+\gamma)\mathrm{b}^2-4(\mathrm{pbx}+\mathrm{aqy})|}}$$
$$|\varepsilon_3| = \frac{(1+\gamma)^2 b^2}{16} + \frac{(1+\gamma)\,|F_1(x,\gamma)|}{12\gamma}.$$

Corollary 3.9. *(i) Let $\hbar \in \Sigma$ given by (1) in the subfamily $\mathfrak{B}^{(\boldsymbol{p,q,x,1})}(\mathrm{y},\mathrm{d};\ \tilde{\mathrm{V}})$ and $\tilde{\mathrm{V}}^{(x,\mathrm{y})}$ be Horadam polynomial with $\mathrm{b}=bx$ and $\mathrm{y}=1$ Then*

$$|\varepsilon_2| \le \frac{(\mathrm{d}+\gamma)|\mathrm{bx}|\sqrt{|\mathrm{bx}|}}{2\sqrt{|(\mathrm{d}+\mathrm{y})\mathrm{b}^2-4(\mathrm{pbx}+\mathrm{aq})|}}$$
$$|\varepsilon_3| = \frac{(\mathrm{d}+\gamma)^2(bx)^2}{16} + \frac{(\mathrm{d}+\gamma)bx}{12\gamma}.$$

(ii) Let $\hbar$ given by (1) be in the subfamily $\mathfrak{B}^{(\boldsymbol{p,q,x,y})}(\gamma,0;\tilde{\mathrm{V}})$. Then

$$|\varepsilon_2| \le \frac{\gamma|\mathrm{bx}|\sqrt{|\mathrm{bx}|}}{2\sqrt{\left|\gamma\mathrm{bx}^2-4\left(\mathrm{pbx}^2+\mathrm{aq}\right)\right|}}$$
$$|\varepsilon_3| = \frac{\gamma^2(bx)^2}{16} + \frac{|\mathrm{bx}|}{12}$$

(iii) Let $\hbar$ given by (1) be in the subfamily $\mathfrak{B}^{(\boldsymbol{p,q,x,y})}(\gamma,1;\tilde{\mathrm{V}})$. Then

$$|\varepsilon_2| \le \frac{(1+\gamma)|\mathrm{bx}|\sqrt{|\mathrm{bx}|}}{2\sqrt{|(1+\gamma)(\mathrm{bx})^2-4\left(\mathrm{pbx}^2+\mathrm{aq}\right)|}}$$
$$|\varepsilon_3| = \frac{(1+\gamma)^2(\mathrm{bx})^2}{16} + \frac{(1+\gamma)\mathrm{bx}}{12\gamma}$$

Corollary 3.10. *(i) Let $\hbar \in \Sigma$ given by (1) in the subfamily $\mathfrak{B}^{(\boldsymbol{p,q,x,1})}(\gamma,\mathrm{d};\tilde{V})$ and $\tilde{\mathrm{V}}^{(x,\mathrm{y})}$ be Chebyshev polynomials with $\mathrm{p}=2,\ \mathrm{b}=2\mathrm{t}, \mathrm{x}=\mathrm{t}, \mathrm{y}=-1, \mathrm{q}=1, \mathrm{a}=1$. Then*

$$|\varepsilon_2| \le \frac{(d+\gamma)|t|\sqrt{|2t|}}{2\sqrt{|(d+\gamma)t^2-(4t^2-1)|}}$$
$$|\varepsilon_3| = \frac{(d+\gamma)^2t^2}{4} + \frac{(d+\mathrm{y})|t|}{6\gamma}$$

(ii) Let $\hbar \in \Sigma$ given by (1) in the subfamily $\mathfrak{B}^{(\mathbf{2,1,t,-1})}(\gamma,0;\tilde{V})$. Then

$$|\varepsilon_2| \le \frac{\gamma|t|\sqrt{|2t|}}{2\sqrt{|\gamma\mathrm{t}^2-(4\mathrm{t}^2-1)|}}$$
$$|\varepsilon_3| \le \frac{\gamma^2\mathrm{t}^2}{4} + \frac{|t|}{6}$$

(iii) Let $\hbar \in \Sigma$ given by (1) in the subfamily $\mathfrak{B}^{(\mathbf{2,1,t,-1})}(\mathrm{y},1;\ \tilde{V})$. Then

$$|\varepsilon_2| \le \frac{(1+\gamma)|t|\sqrt{|2t|}}{\sqrt{|(1+\gamma)4t^2-4\left(2t^2-1\right)|}}$$
$$|\varepsilon_3| \le \frac{(1+\gamma)^2t^2}{4} + \frac{(1+\gamma)|t|}{6\gamma}$$

Corollary 3.11. *(i) Let* $\hbar \in \Sigma$ *given by* (1) *in the subfamily* $\mathfrak{B}^{(1,1,x,1)}(\gamma, d; \tilde{V})\mathrm{y}$ *and* $\tilde{\mathrm{V}}^{(x,\mathrm{y})}$ *be Fibonacci polynomials with* $\mathrm{p} = \mathrm{q} = 1,\ \mathrm{b} = 1, \mathrm{y} = 1, \mathrm{a} = 0$. *Then*

$$|\varepsilon_2| \leq \frac{(d+\gamma)}{2\sqrt{|(d+\gamma) - 4x|}}$$
$$|\varepsilon_3| \leq \frac{(d+\gamma)^2}{16} + \frac{(d+\gamma)}{12\gamma}$$

(ii) Let $\hbar \in \Sigma$ *given by* (1) *in the subfamily* $\mathfrak{B}^{(\mathbf{1},\mathbf{1},\mathbf{x},\mathbf{1})}(\gamma, 0; \tilde{\mathrm{V}})$. *Then*

$$|\varepsilon_2| \leq \frac{\gamma}{2\sqrt{|\gamma - 4x|}}$$
$$|\varepsilon_3| \leq \frac{\gamma^2}{16} + \frac{\mathbf{1}}{12}$$

(iii) Let $\hbar \in \Sigma$ *given by* (1) *in the subfamily* $\mathfrak{B}^{(\mathbf{1},\mathbf{1},\boldsymbol{x},\mathbf{1})}(\gamma, 1; \tilde{V})$. *Then*

$$|\varepsilon_2| \leq \frac{(1+\mathrm{y})}{2\sqrt{|(1+\gamma) - 4\mathrm{x}|}}$$
$$|\varepsilon_3| \leq \frac{(1+\gamma)^2}{16} + \frac{(1+\gamma)}{12\gamma}$$

4. Subordination and Applications

The unseen body's reaction to electromagnetic radiation has been tackled recently in RCS studies. The scientific community has witnessed a growing interest in Electromagnetic cloaking, particularly among those investigators who improve so named met materials-artificial composites that contain striking electromagnetic features. From mathematical perspective, both the cloaked object and the two-dimensional cloak are possible to be regarded as simple linked districts in complex level. The two districts are equal to conformal maps that relate to unit circle as it stated in Riemann Mapping Theorem. Let us consider the function $g(z)$ symbolizes the cloaked object while the function $\mathrm{q}(z)$ denotes the cloak. Then, we obtain the following:

$$g(z) \prec \mathrm{q(z)}.$$

As the cloak relies on unseen body as well as the rays echoed by the body, a cloak may exist for not comprising all mirrored rays. Accordingly, it is enhanced to be a three-dimensional cloak.

5. Discussion of the results

This work presents the upper bound of the initial coefficient, and the possibility of these coefficients that affect the ranges of functions in the following new families $\mathcal{M}^{p,q,x,y}(\varrho, \xi, \Sigma; \tilde{\mathrm{V}})$ and $\mathfrak{B}^{(p,q,x,\mathrm{y})}(\gamma, \mathrm{d}; \tilde{\mathrm{V}})$. In this regard, the two-dimensional cloak and the covered object can be regarded as simple connected regions in a complex plane. Through considering the parameters in Theorems 2.7 and 3.7, various interesting as well as particular cases of the main theorems are highlighted in the form of corollaries.

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