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Decay Solutions of Coupled Schrödinger Equation with Internal Fractional Damping

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Abstract

In this work, we study a coupled Schrödinger equation with an internal fractional damping. First, we reformulate the system into an augmented model and we establish the existence of the solutions through the theory of semigroup. Then, we prove the strong stability using the theorem of Arendt-Batty. A polynomial decay of the energy is shown by applying the theorem of A. Borichev and Y. Tomilov. Finally, we show the optimality decay by proving the lack of exponential stability.

Keywords: Coupled Schrödinger equation, Internal fractional damping, semigroup theory, polynomial stability.
2010 MSC: 35B40, 35Q41, 93D15.

1. Introduction

In this paper, we are concerned with the existence and decay properties of solutions for the problem of a coupled Schrödinger equation of the type

$$\begin{cases} iu_t(x,t) + \Delta u(x,t) + i\gamma \partial^{\alpha,\eta} u(x,t) + i\beta v(x,t) = 0, & \text{in } \Omega \times \mathbb{R}_+, \\ iv_t(x,t) + \Delta v(x,t) - i\beta u(x,t) = 0, & \text{in } \Omega \times \mathbb{R}_+, \end{cases} \tag{1}$$

where Ω is an open bounded domain of $\mathbb{R}^n$ with sufficiently smooth boundary, $\beta > 0$ and the term $\partial^{\alpha,\eta}$ stands for the generalized Caputo's fractional derivative of order α with respect to the time variable (see [8]), which is defined by

$$\partial^{\alpha,\eta} w(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} w(s)\, ds, \quad 0 < \alpha < 1, \quad \eta \geq 0,$$

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where Γ denotes the Gamma function.
The system is subject to the boundary conditions

$$u(x,t) = v(x,t) = 0, \quad \text{in } \partial\Omega \times \mathbb{R}_+^*.$$

This system is finally completed with the initial conditions

$$u(x,0) = u_0(x)$$

and

$$v(x,0) = v_0(x),$$

where the initial data u_0 and v_0 belongs to a suitable function space.

This work is inspired by recent work published in [13].
Systems of coupled second order differential equations arise in many problems in molecular quantum physics.

Several authors studied the Schrödinger problems in bounded or unbounded domains.
In [10], Lin et al. treated the nonlinear problem

$$\begin{cases} -\Delta u + (\mu V_1(x) + a)u = f(x)|u|^{p-2}u + \beta(x)v, & x \in \mathbb{R}^N, \\ -\Delta v + (\mu V_2(x) + b)v = g(x)|v|^{p-2}v + \beta(x)u, & x \in \mathbb{R}^N, \end{cases}$$

where $N \geq 3$, $2 < p \leq 2$, $a, b \in \mathbb{R}$ such that $a > -\lambda_1(\Omega_1)$, $b > -\lambda_1(\Omega_2)$ and $\lambda_1(\Omega_i)$ is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega_i)$, $\mu > 0$, V_1 and V_2 are potentials wells with bottoms $\Omega_i = int\ V_i^{-1}(0)$, the weight functions f and g are continuous and nonnegative in $\mathbb{R}^N$, and the coupling function β is related to the potentials V_1, V_2 and parameters a, b. The authors obtained some results of the existence of positive ground states to a linear coupled Schrödinger systems in a bounded domain.

Very recently, In [5], Bhandari et al., studied the well-posedness and the controllability in a bounded domain in $\mathbb{R}$, more precisely, in $\Omega = (0.1)$ of the problem

$$\begin{cases} iu_t(t,x) + \gamma_1 u_{xx}(t,x) - \alpha_1 u(x,t) = 0, & \text{in } (0,T) \times \Omega, \\ i\sigma v_t(t,x) + \gamma_2 v_{xx}(t,x) = 0, & \text{in } (0,T) \times \Omega, \\ u(0,x) = u_0(x),\ v(0,x) = v_0(x), & \text{in } \Omega, \end{cases}$$

where the constants $\sigma, \gamma_1, \gamma_1, \alpha_1, \alpha_2$ are positive and (u_0, v_0) are given initial data in certain spaces. Their motivation comes from the following system

$$\begin{cases} iu_t(t,x) + pu_{xx}(t,x) - \theta u(x,t) + \widetilde{u}(t,x)v(t,x) = 0, & t \geq 0,\ x \in \mathbb{R}, \\ i\sigma v_t(t,x) + qv_{xx}(t,x) - \varrho v(t,x) + \dfrac{a}{2}u^2(t,x) = 0, \\ u(0,x) = u_0(x),\ v(0,x) = v_0(x), \end{cases}$$

where u and v are complex-valued functions and θ, ϱ and $a := \dfrac{1}{\sigma}$ are real numbers representing physical parameters of the system, where $\sigma > 0$ and $p, q = 1$ or $p, q = -1$.

Recently, in [16], Zhang studied the stability of a Schrödinger system with one boundary damping, which consists of two constant coefficients Schrödinger equations coupled through zero-order terms which describes molecular multiphoton transitions caused by a laser [3]. The system is given by

$$\begin{cases} iy_t + \rho\Delta y + \alpha z = 0, & \text{in } \Omega\times(0,+\infty), \\ iz_t + \Delta z + \alpha y = 0, & \text{in } \Omega\times(0,+\infty), \\ y = 0, & \text{on } \Gamma_1\times(0,+\infty), \\ \dfrac{\partial y}{\partial \nu} = iy, & \text{on } \Gamma_2\times(0,+\infty), \\ z = 0, & \text{in } \Gamma\times(0,+\infty), \\ y(0) = y_0, z(0) = z_0, & \text{in } \Omega, \end{cases}$$

where ρ is positive constant, $\alpha \in \mathbb{R}^*$, $\Omega \subset \mathbb{R}^n (n \in \mathbb{N}^+)$ be a bounded domain with $\mathcal{C}^2$ boundary $\partial\Omega = \Gamma$ and ν is the unit outward normal vector of Γ. Assume that $\Gamma = \Gamma_1 \cup \Gamma_1$, $\bar{\Gamma}_1 \cap \bar{\Gamma}_2 = \emptyset$, $\Gamma_1 \neq \emptyset$ and $\Gamma_2 \neq \emptyset$. In first time, the author used the asymptotic expansions of eigenvalues to show that the system is not exponentially stable when $\rho = 0$, and he proved that the energy decay rate of the multidimensional Schrödinger system is t^{-1} for sufficiently smooth initial data when $\rho = 1$, $|\alpha|$ is sufficiently small, and the boundary of domain satisfies suitable geometric assumption using the frequency domain approach and the multiplier method. Next, he showed that the strong stability of the system is determined by ρ and α. Finally, by using the frequency domain approach and solving the resolvent equation of unbounded operator, he proved that the energy of the system decays polynomially and the decay rate depends on the arithmetic property of ρ when $\rho \neq 1$.

Moreover, in [9], Hamchi et al. considered a coupled system of two complex Schrödinger equations with variable coefficients where the boundary feedback appears only in one of the equations. They considered the same geometric assumptions as in [16], the system is given by

$$\begin{cases} iy_t + Ay + az = 0, & \text{in } \Omega\times(0,+\infty), \\ iz_t + Az + ay = 0, & \text{in } \Omega\times(0,+\infty), \\ y = 0, & \text{on } \Gamma_1\times(0,+\infty), \\ \dfrac{\partial y}{\partial \nu_A} + by_t = 0, & \text{on } \Gamma_2\times(0,+\infty), \\ z = 0, & \text{in } \Gamma\times(0,+\infty), \\ y(0) = y_0, z(0) = z_0, & \text{in } \Omega, \end{cases}$$

where A the operator defined by

$$Ay = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial y}{\partial x_j}\right),$$

it is a second order differential operator with real coefficients $a_{ij} = a_{ji}$ of class $\mathcal{C}^1$ and satisfies the uniform ellipticity condition.
a and b are two functions in $L^\infty(\Omega)$ such that for some constants $a_*, b_* > 0$, we have

$$a_* \leq a(x), \quad \forall\, x \in \Omega, \quad \text{and} \quad b_* \leq \Re(b(x)) = b(x), \quad \forall\, x \in \Gamma.$$

This type of system describes a particle in a box with two levels internal and subject to a wave resonant laser with the transition between the two energy levels [15].
The authors proved that they can apply the Riemann geometric approach to the coupled complex Schrödinger equations with variables coefficients and by adapting the method of Alabau developed in the context of coupled real wave equations with constant coefficients used in [1], they can get the indirect boundary stabilization of the system.

This paper is organized as follow: In Section 2, we present some preliminary results and we reformulate the system given into an augmented system. In Section 3, we prove the well-posedness of the system by semigroup theory. In Section 4, we prove an asymptotic and polynomial decay. First, we prove the strong stability using the theorem of Arendt-Batty [2]. Next, we establish the polynomial decay of the energy by the theorem of A. Borichev and Y. Tomilov [6]. In Section 5, we show the optimality decay by proving the lack of exponential stability of solutions.

2. Preliminary results

2.1. Augmented model

In this section we reformulate (1) into an augmented system. For that, we need the following auxiliary results.

Theorem 2.1 (see [12]). *Let μ be the function:*

$$\mu(\xi) = |\xi|^{(2\alpha-1)/2}, \quad -\infty < \xi < +\infty,\ 0 < \alpha < 1. \tag{2}$$

μ is an even non negative measurable function verifying

$$\int_{-\infty}^{+\infty} \frac{\mu(\xi)^2}{1+\xi^2} d\xi < +\infty. \tag{3}$$

The relationship between the 'input' U and the 'output' O of the system

$$\partial_t\theta(x,\xi,t) + \xi^2\theta(x,\xi,t) + \eta\theta(x,\xi,t) - U(t)\mu(\xi) = 0, \quad -\infty < \xi < +\infty, \eta \geq 0, t > 0, \tag{4}$$

$$\theta(x,\xi,0) = 0, \tag{5}$$

$$O(t) = (\pi)^{-1}\sin(\alpha\pi)\int_{-\infty}^{+\infty} \mu(\xi)\theta(x,\xi)\, d\xi, \tag{6}$$

is given by

$$O(t) = I^{1-\alpha,\eta}U(t), \tag{7}$$

where

$$[I^{\alpha,\eta}f](t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}e^{-\eta(t-\tau)}f(\tau)\, d\tau.$$

Here, taking the input $U(x,t) = u(x,t)$, then combining (1) with (7), we obtain

$$O(t) = I^{1-\alpha,\eta}u(x,t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}e^{-\eta(t-s)}u(x,s)\, ds = \partial^{\alpha,\eta}u(x,t).$$

By substituting this equality into theorem 2.1, we get

$$\begin{cases} \partial_t\theta(x,\xi,t) + (\xi^2+\eta)\theta(x,\xi,t) - U(t)\mu(\xi) = 0, & (x,\xi,t) \in \Omega \times \mathbb{R} \times \mathbb{R}^+, \\ \theta(x,\xi,0) = 0, & (x,\xi) \in \Omega \times \mathbb{R}, \\ \partial^{\alpha,\eta}u(x,t) - (\pi)^{-1}\sin(\alpha\pi)\displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\theta(x,\xi)\, d\xi = 0, & (x,\xi,t) \in \Omega \times \mathbb{R}^+. \end{cases} \tag{8}$$

Using now the representation (8), system (1) can be written as the augmented model:

$$\begin{cases} u_t(x,t) - i\Delta u(x,t) + \tilde{\gamma}\displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\theta(x,\xi,t)\, d\xi + \beta v(x,t) = 0, & x \in \Omega,\ t > 0, \\ v_t(x,t) - i\Delta v(x,t) - \beta u(x,t) = 0, & x \in \Omega,\ t > 0, \\ \partial_t\theta(x,\xi,t) + (\xi^2+\eta)\theta(x,\xi,t) - \mu(\xi)u = 0, & x \in \Omega,\ \xi \in \mathbb{R},\ t > 0, \\ u(x,t) = 0, v(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), v(x,0) = v_0(x), & x \in \Omega, \\ \theta(x,\xi,0) = 0, & x \in \Omega,\ \xi \in \mathbb{R}, \end{cases} \tag{9}$$

with $\widetilde{\gamma} = \gamma\pi^{-1}\sin(\alpha\pi)$.

We define the energy associated to the solution of the problem (9) by the following formula

$$\mathcal{E}(t) = \frac{1}{2}\|u\|^2_{L^2(\Omega)} + \frac{1}{2}\|v\|^2_{L^2(\Omega)} + \frac{\widetilde{\gamma}}{2}\|\theta\|^2_{L^2(\Omega\times(-\infty,+\infty))}. \tag{10}$$

Lemma 2.2. *Let (u, v, θ) be a regular solution of the problem (9), then the energy $\mathcal{E}(t)$ satisfies*

$$\frac{d}{dt}\mathcal{E}(t) = -\widetilde{\gamma}\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\eta)|\theta(x,\xi,t)|^2 d\xi dx \leq 0. \tag{11}$$

Proof. Multiplying the first equation of (9) by $\bar{u}$ and the second one by $\bar{v}$, integrating over Ω and multiplying the third one by $\widetilde{\gamma}\bar{\theta}$, integrating over $\Omega\times(-\infty,+\infty)$ and applying the Green formula we get the result.

We shall need the following lemma in all sections.

Lemma 2.3 (see [4]). *If $\lambda \in D_\eta = \mathbb{C}\backslash]-\infty,-\eta]$ then*

$$\int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{\lambda+\eta+\xi^2}d\xi = \frac{\pi}{\sin\alpha\pi}(\lambda+\eta)^{\alpha-1}.$$

3. The well-posedness of the problem

This section is concerned to the well-posedness results of the problem (9) using a semigroup approach and the Lumer-Philips theorem.
Introducing the vector function $Y = (u, v, \theta)^T$, then system (9) is equivalent to

$$\begin{cases} \partial_t Y = \mathcal{A}Y, \\ Y(0) = Y_0 = (u_0, v_0, \theta_0)^T, \end{cases} \tag{12}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}Y = \left(i\Delta u - \widetilde{\gamma}\int_{-\infty}^{+\infty}\mu(\xi)\theta(x,\xi)d\xi - \beta v, i\Delta v + \beta u, -(\xi^2+\eta)\theta(x,\xi) + \mu(\xi)u\right)^T, \tag{13}$$

with domain

$$D(\mathcal{A}) = \left\{\begin{array}{l} (u,v,\theta) \text{ in } \mathcal{H} : u, v \in H^2(\Omega)\cap H^1_0(\Omega), \\ i\Delta u - \widetilde{\gamma}\int_{-\infty}^{+\infty}\mu(\xi)\theta(x,\xi)d\xi \in L^2(\Omega), \\ -\left(\xi^2+\eta\right)\theta + u\mu(\xi) \in L^2(\Omega\times(-\infty,+\infty)), \\ |\xi|\theta \in L^2(\Omega\times(-\infty,+\infty)), \end{array}\right\} \tag{14}$$

where

$$\mathcal{H} = L^2(\Omega)\times L^2(\Omega)\times L^2(\Omega\times(-\infty,+\infty)),$$

equipped with the inner product

$$\langle Y, \tilde{Y}\rangle_{\mathcal{H}} = \int_\Omega u\bar{\tilde{u}}\,dx + \int_\Omega v\bar{\tilde{v}}\,dx + \tilde{\gamma}\int_\Omega\int_{-\infty}^{+\infty}\theta\bar{\tilde{\theta}}d\xi dx,$$

for $Y = (u, v, \theta), \tilde{Y} = (\tilde{u}, \tilde{v}, \tilde{\theta}) \in \mathcal{H}$.

The main result in this section is given by the following theorem

Theorem 3.1. *(1) If $Y_0 \in D(\mathcal{A})$, then system (12) has a unique strong solution*

$$Y \in C^0(\mathbb{R}_+, D(\mathcal{A})) \cap C^1(\mathbb{R}_+, \mathcal{H}).$$

(2) If $Y_0 \in \mathcal{H}$, then system (12) has a unique weak solution

$$Y \in C^0(\mathbb{R}_+, \mathcal{H}).$$

Proof. To prove this result we shall use the semigroup approach, in particular Lumer-Phillips theorem. Since for every $Y \in D(\mathcal{A})$, we have

$$\Re\langle \mathcal{A}Y, Y\rangle_{\mathcal{H}} = -\widetilde{\gamma}\int_\Omega \int_{-\infty}^{+\infty} (\xi^2+\eta)|\theta(x,\xi)|^2\, d\xi dx < 0. \tag{15}$$

Then $\mathcal{A}$ is dissipative.
Next, we prove that the operator $I - \mathcal{A}$ is surjective. For this purpose, let us take $F = (f_1, f_2, f_3)^T \in \mathcal{H}$, we prove that there exists $Y \in D(\mathcal{A})$ satisfying

$$Y - \mathcal{A}Y = F. \tag{16}$$

Equation (16) is equivalent to

$$\begin{cases} u - i\Delta u + \widetilde{\gamma}\displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\theta(x,\xi)d\xi + \beta v = f_1, \\ v - i\Delta v - \beta u = f_2, \\ \theta + (\xi^2+\eta)\theta - \mu(\xi)u = f_3. \end{cases} \tag{17}$$

Using $(17)_3$, we obtain

$$\theta(x,\xi) = \frac{f_3(x,\xi) + \mu(\xi)u}{\xi^2+\eta+1}. \tag{18}$$

By substituting (18) into $(17)_1$, we get

$$u - i\Delta u + \widetilde{\gamma}\int_{-\infty}^{+\infty} \frac{u}{\xi^2+\eta+1}\mu^2(\xi)d\xi + \beta v = f_1 - \widetilde{\gamma}\int_{-\infty}^{+\infty}\frac{f_3(x,\xi)}{\xi^2+\eta+1}\mu(\xi)d\xi, \tag{19}$$

then, the variational formulation corresponding to (17) is

$$\mathcal{B}(\psi, w) = \mathcal{L}(w), \tag{20}$$

with $\psi = (u, v)$, $w = (w_1, w_2) \in (H_0^1(\Omega))^2$ and $\mathcal{B}$ is the sesquilinear form given by

$$\begin{aligned} \mathcal{B}(\psi, w) &= \int_\Omega u\overline{w}_1 + \int_\Omega v\overline{w}_2 - i\int_\Omega \Delta u\overline{w}_1 - i\int_\Omega \Delta v\overline{w}_2 - \beta\int_\Omega u\overline{w}_2 + \beta\int_\Omega v\overline{w}_1 \\ &+ \widetilde{\gamma}\int_\Omega\int_{-\infty}^{+\infty}\frac{u\mu^2(\xi)}{\xi^2+\eta+1}\overline{w}_1 d\xi dx, \end{aligned}$$

and $\mathcal{L}$ is the antilinear functional defined by

$$\mathcal{L}(w) = \int_\Omega f_1\overline{w}_1\, dx - \widetilde{\gamma}\int_\Omega \overline{w}_1\int_{-\infty}^{+\infty}\frac{f_3(x,\xi)\mu(\xi)}{\xi^2+\eta+1}d\xi dx + \int_\Omega f_2\overline{w}_2,$$

It is not hard to verify that $\mathcal{B}$ is continuous and coercive and $\mathcal{L}$ is continuous. Consequently, by the Lax-Milgram theorem, system (20) has a unique solution $(u, v) \in (H_0^1(\Omega))^2$, for all $w \in (H_0^1(\Omega))^2$. Applying the classical elliptic regularity, it follows that $(u, v) \in (H^2(\Omega))^2$. Therefore, the operator $I - \mathcal{A}$ is surjective.

4. Asymptotic behavior

4.1. Strong stability of the system

This section is devoted to study the strong stability of solution associated with the problem (12). For this aim we apply a version of the Arendt-Batty and Lyubich-Vu for Hilbert spaces [2], [11].

Theorem 4.1 ([2], [11]). *Let $\mathcal{A}$ be the generator of a uniformly bounded C_0-semigroup $\{S(t)\}_{t\geq 0}$ on a Hilbert space $\mathcal{H}$. If:*

(i) $\mathcal{A}$ does not have eigenvalues on $i\mathbb{R}$.

(ii) The intersection of the spectrum $\sigma(\mathcal{A})$ with $i\mathbb{R}$ is at most a countable set,

then the semigroup $\{S(t)\}_{t\geq 0}$ is asymptotically stable, i.e, $\|S(t)z\|_{\mathcal{H}} \to 0$ as $t \to \infty$ for any $z \in \mathcal{H}$.

Our next main result in this part is the following theorem.

Theorem 4.2. *The C_0-semigroup $\{S(t)\}_{t\geq 0}$ is strongly stable in $\mathcal{H}$, i.e, for all $Y_0 \in \mathcal{H}$, the solution of (12) satisfies*

$$\lim_{t\to\infty} \|S(t)Y_0\|_{\mathcal{H}} = 0.$$

In order to prove Theorem 4.2, we need the following two lemmas.

Lemma 4.3. *For all $\lambda \in \mathbb{R}$, we have $i\lambda I - \mathcal{A}$ is injective, that is*

$$Ker(i\lambda I - \mathcal{A}) = \{0\}.$$

Proof. Let $\lambda \in \mathbb{R}$ be such that $i\lambda$ is an eigenvalue of the operator $\mathcal{A}$ and let $Y = (u, v, \theta) \in D(\mathcal{A})$ be a corresponding eigenvector such that

$$\mathcal{A}Y = i\lambda Y. \tag{21}$$

Equivalently,

$$\begin{cases} i\lambda u - i\Delta u + \widetilde{\gamma}\displaystyle\int_{-\infty}^{+\infty} \theta(x,\xi)\mu(\xi)d\xi + \beta v = 0, \\ i\lambda v - i\Delta v - \beta u = 0, \\ i\lambda\theta + (\xi^2+\eta)\theta - \mu(\xi)u = 0. \end{cases} \tag{22}$$

From (15) and (21), we obtain

$$0 = \Re\langle \mathcal{A}Y, Y\rangle_{\mathcal{H}} = -\widetilde{\gamma}\int_\Omega \int_{-\infty}^{+\infty} (\xi^2+\eta)|\theta(x,\xi)|^2\, d\xi dx.$$

It's clear that

$$\theta(x,\xi) = 0 \ \text{ i.e. in } \ \Omega \times (-\infty, +\infty). \tag{23}$$

By substituting (23) into $(22)_3$, we obtain

$$u = 0,$$

so

$$v = 0,$$

it follows that $Y = 0$.

Lemma 4.4. *If $\eta > 0$ and $\lambda \in \mathbb{R}$ or $\eta = 0$ and $\lambda \in \mathbb{R}^*$, then $i\lambda I - \mathcal{A}$ is surjective.*

Proof.
Case 1: $\lambda \neq 0$. Let $F = (f_1, f_2, f_3)^T \in \mathcal{H}$ be given. We search for $Y = (u, v, \theta)^T \in D(\mathcal{A})$ such that

$$(i\lambda I - \mathcal{A})Y = F.$$

Equivalently, we have

$$\begin{cases} i\lambda u - i\Delta u + \widetilde{\gamma}\displaystyle\int_{-\infty}^{+\infty} \theta(x,\xi)\mu(\xi)d\xi + \beta v = f_1, \\ i\lambda v - i\Delta v - \beta u = f_2, \\ i\lambda\theta + (\xi^2+\eta)\theta - \mu(\xi)u = f_3. \end{cases} \tag{24}$$

Using $(24)_3$ we can find θ as

$$\theta(x,\xi) = \frac{f_3(x,\xi) + \mu(\xi)u}{\xi^2+\eta+i\lambda}. \tag{25}$$

By substituting (25) in $(24)_1$, we obtain

$$i\lambda u - i\Delta u + \widetilde{\gamma}\int_{-\infty}^{+\infty} \frac{u}{\xi^2+\eta+i\lambda}\mu^2(\xi)d\xi + \beta v = f_1 - \widetilde{\gamma}\int_{-\infty}^{+\infty}\frac{f_3(x,\xi)}{\xi^2+\eta+i\lambda}\mu(\xi)d\xi, \tag{26}$$

Solving system (26) is equivalent to finding $u, v \in H^2(\Omega)\cap H_0^1(\Omega)$ such that

$$\lambda\int_\Omega u\overline{w}_1 + \lambda\int_\Omega v\overline{w}_2 - \int_\Omega \Delta u\overline{w}_1 - \int_\Omega \Delta v\overline{w}_2 + i\beta\int_\Omega u\overline{w}_2 - i\beta\int_\Omega v\overline{w}_1 - i\widetilde{\gamma}\int_\Omega\int_{-\infty}^{+\infty}\frac{u\mu^2(\xi)}{\xi^2+\eta+i\lambda}\overline{w}_1 d\xi dx$$
$$= -i\int_\Omega f_1\overline{w}_1\,dx + i\widetilde{\gamma}\int_\Omega \overline{w}_1\int_{-\infty}^{+\infty}\frac{f_3(x,\xi)\mu(\xi)}{\xi^2+\eta+i\lambda}d\xi dx - i\int_\Omega f_2\overline{w}_2,$$

for all $w_1, w_2 \in H_0^1(\Omega)$.

Then, the variational formulation corresponding to (26) is

$$\widetilde{\mathcal{B}}(\psi, w) = \widetilde{\mathcal{L}}(w), \tag{27}$$

where the sesquilinear form $\widetilde{\mathcal{B}}$ and the antilinear form $\widetilde{\mathcal{L}}$ are defined by

$$\begin{aligned}\widetilde{\mathcal{B}}(\psi,w) &= \lambda\int_\Omega u\overline{w}_1 + \lambda\int_\Omega v\overline{w}_2 - \int_\Omega \Delta u\overline{w}_1 - \int_\Omega \Delta v\overline{w}_2 + i\beta\int_\Omega u\overline{w}_2 - i\beta\int_\Omega v\overline{w}_1 \\ &- i\widetilde{\gamma}\int_\Omega\int_{-\infty}^{+\infty}\frac{u\mu^2(\xi)}{\xi^2+\eta+i\lambda}\overline{w}_1 d\xi dx\end{aligned}$$

and

$$\widetilde{\mathcal{L}}(w) = -i\int_\Omega f_1\overline{w}_1\,dx + i\widetilde{\gamma}\int_\Omega \overline{w}_1\int_{-\infty}^{+\infty}\frac{f_3(x,\xi)\mu(\xi)}{\xi^2+\eta+i\lambda}d\xi dx - i\int_\Omega f_2\overline{w}_2,$$

with $\psi = (u, v)$, $w = (w_1, w_2)$.
It is easy to verify that $\widetilde{\mathcal{B}}$ is continuous and coercive, and $\widetilde{\mathcal{L}}$ is continuous. Owing to the Lax-Milgram theorem, we conclude that for all $w \in (H_0^1(\Omega))^2$ the problem (27) admits a unique solution $(u, v) \in (H_0^1(\Omega))^2$. Applying the classical elliptic regularity, it follows that $(u, v) \in (H^2(\Omega))^2$. Therefore, the operator $i\lambda I - \mathcal{A}$ is surjective.

Case 2: $\lambda = 0$ and $\eta \neq 0$. In this case the system (24) is reduced to the following

$$\begin{cases} -i\Delta u + \widetilde{\gamma}\displaystyle\int_{-\infty}^{+\infty} \theta(x,\xi)\mu(\xi)d\xi + \beta v = f_1, \\ -i\Delta v - \beta u = f_2, \\ (\xi^2+\eta)\theta(\xi,x) - \mu(\xi)u = f_3. \end{cases} \tag{28}$$

Using $(28)_3$, we get

$$\theta(x,\xi) = \frac{f_3(x,\xi) + \mu(\xi)u}{\xi^2+\eta}. \tag{29}$$

From (29), we obtain

$$-i\Delta u + \widetilde{\gamma}\int_{-\infty}^{+\infty} \frac{u}{\xi^2+\eta}\mu^2(\xi)d\xi + \beta v = f_1 - \widetilde{\gamma}\int_{-\infty}^{+\infty} \frac{f_3(x,\xi)}{\xi^2+\eta}\mu(\xi)d\xi. \tag{30}$$

Solving system (30) is equivalent to finding $u, v \in H^2(\Omega)\cap H_0^1(\Omega)$ such that

$$\begin{aligned} -i\int_\Omega \Delta u\overline{w}_1 - i\int_\Omega \Delta v\overline{w}_2 - \beta\int_\Omega u\overline{w}_2 + \beta\int_\Omega v\overline{w}_1 + \widetilde{\gamma}\int_\Omega\int_{-\infty}^{+\infty}\frac{u\mu^2(\xi)}{\xi^2+\eta}\overline{w}_1 d\xi dx \\ = \int_\Omega f_1\overline{w}_1\, dx - \widetilde{\gamma}\int_\Omega \overline{w}_1\int_{-\infty}^{+\infty}\frac{f_3(x,\xi)\mu(\xi)}{\xi^2+\eta}d\xi dx + \int_\Omega f_2\overline{w}_2, \end{aligned}$$

for all $w_1, w_2 \in H_0^1(\Omega)$.
Then, we get the following variational formulation

$$\mathcal{B}'(u,w) = \mathcal{L}'(w) \tag{31}$$

where the sesquilinear form $\mathcal{B}'$ and the antilinear form $\mathcal{L}'$ are defined by

$$\mathcal{B}'(\psi,w) = -\int_\Omega \Delta u\overline{w}_1 - \int_\Omega \Delta v\overline{w}_2 + i\beta\int_\Omega u\overline{w}_2 - i\beta\int_\Omega v\overline{w}_1 - i\widetilde{\gamma}\int_\Omega\int_{-\infty}^{+\infty}\frac{u\mu^2(\xi)}{\xi^2+\eta}\overline{w}_1 d\xi dx$$

and

$$\mathcal{L}'(w) \;=\; -i\int_\Omega f_1\overline{w}_1\, dx + i\widetilde{\gamma}\int_\Omega \overline{w}_1\int_{-\infty}^{+\infty}\frac{f_3(x,\xi)\mu(\xi)}{\xi^2+\eta}d\xi dx - i\int_\Omega f_2\overline{w}_2,$$

with $\psi = (u,v)$, $w = (w_1,w_2)$.
It is not hard to verify that $\mathcal{B}'$ is continuous and coercive, and $\mathcal{L}'$ is continuous. Using the Lax-Milgram theorem, we deduce that for all $w \in (H_0^1(\Omega))^2$ the problem (31) admits a unique solution $(u,v) \in (H_0^1(\Omega))^2$. By the regularity theory for the linear elliptic equations, it follows that $(u,v) \in (H^2(\Omega))^2$. Then, the operator $i\lambda I - \mathcal{A}$ is surjective.

According to lemmas 4.3, 4.4 and Theorem 4.1 the C_0−semigroup $\{S(t)\}_{t\geq 0}$ is strongly stable in $\mathcal{H}$.

4.2. Polynomial stability

In this section we will prove a polynomial decay rate, our main result is the following theorem.

Lemma 4.5. *Assume that $\mathcal{A}$ is the generator of a strongly continuous semigroup of contractions $\{S(t)\}_{t\geq 0}$ on a Hilbert space $\mathcal{H}$. If*

$$i\mathbb{R} \subset \rho(\mathcal{A}), \tag{32}$$

then for a fixed $l > 0$, the following conditions are equivalent:

$$\lim_{|\lambda|\to\infty}\sup_{\lambda\in\mathbb{R}} \frac{1}{|\lambda|^l}\|(i\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty, \tag{33}$$

$$\|S(t)Y_0\|_{\mathcal{H}}^2 \leq \frac{c}{t^{\frac{2}{l}}}\|Y_0\|_{D(\mathcal{A})}^2,\ Y_0 \in D(\mathcal{A}),\quad \textit{for some } C > 0. \tag{34}$$

Theorem 4.6. *The semigroup* $\{S(t)\}_{t\geq 0}$ *is polynomially stable and*

$$E(t) = \|S(t)Y_0\|^2_{\mathcal{H}} \leq \frac{1}{t^{\frac{2}{(1-\alpha)}}}\|Y_0\|^2_{D(\mathcal{A})}. \tag{35}$$

Proof. Suppose that the condition (33) is not satisfied. By an argument of contradiction, we suppose that there exist a sequence $\lambda_n \in \mathbb{R}$ and $\lambda_n \longrightarrow +\infty$ if $n \longrightarrow \infty$ and a sequence $Y_n = (u_n, v_n, \theta_n) \in D(\mathcal{A})$ such that

$$\|Y_n\| = 1, \tag{36}$$

and

$$\lim_{n\to\infty} \frac{1}{\lambda_n^l}\|(i\lambda_n I - \mathcal{A})^{-1}\|_{\mathcal{L}(H)} = +\infty. \tag{37}$$

For simplification, we denote λ_n by λ and Y_n by $Y = (u, v, \theta)$ and

$$F_n = \lambda_n^l(i\lambda_n I - A)Y_n = (f_{1_n}, f_{2_n}, f_{3_n}),$$

by

$$F = \lambda^l(i\lambda I - A)Y = (f_1, f_2, f_3).$$

From (37), we obtain

$$\begin{cases} i\lambda u - i\Delta u + \widetilde{\gamma}\displaystyle\int_{-\infty}^{+\infty}\theta(x,\xi,t)\mu(\xi)d\xi + \beta v = \frac{f_1}{\lambda^l} \to 0, \text{ in } L^2(\Omega) \\ i\lambda v - i\Delta v - \beta u = \dfrac{f_2}{\lambda^l} \to 0, \text{ in } L^2(\Omega) \\ i\lambda\theta + (\xi^2+\eta)\theta - \mu(\xi)u = \dfrac{f_3}{\lambda^l} \to 0 \text{ in } L^2((\Omega)\times]-\infty,+\infty[) \end{cases} \tag{38}$$

We need the following results.

Lemma 4.7. *Under (38) we have*

$$\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\eta)|\theta(x,\xi)|^2\,d\xi dx = \frac{o(1)}{\lambda^l}, \tag{39}$$

$$\int_\Omega\int_{-\infty}^{+\infty}|\theta(x,\xi)|^2\,d\xi dx = \frac{o(1)}{\lambda^l}. \tag{40}$$

and

$$\int_\Omega\left|\int_{-\infty}^{+\infty}\mu(\xi)\theta(x,\xi)\,d\xi\right|^2 dx = \frac{o(1)}{\lambda^l}. \tag{41}$$

Proof. From (15) and (37), we have

$$\Re\langle i\lambda Y - \mathcal{A}Y, Y\rangle_{\mathcal{H}} = \widetilde{\gamma}\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\eta)|\theta(x,\xi)|^2\,d\xi dx = \frac{o(1)}{\lambda^l},$$

which implies (39).

The estimation (40) is a consequence of

$$\int_\Omega\int_{-\infty}^{+\infty}|\theta(x,\xi)|^2\,d\xi dx \quad\leq\quad \int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\eta)|\theta(x,\xi)|^2\,d\xi dx.$$

Since $Y \in D(\mathcal{A})$, by using Cauchy-Schwarz's inequality and (3), we have

$$\int_\Omega\left|\int_{-\infty}^{+\infty}\mu(\xi)\theta(x,\xi)\,d\xi\right|^2 dx \leq C\left(\int_{-\infty}^{+\infty}|(\xi^2+\eta)^{-1}|\mu(\xi)|^2d\xi\right)\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\eta)|\theta(x,\xi)|^2d\xi.$$

This proves (41).

Lemma 4.8. *We have*

$$\int_\Omega |u(x)|^2 dx = \frac{o(1)}{\lambda^{l+\alpha-1}}.$$

Proof. From (38)$_3$, we have

$$(i\lambda + \xi^2 + \eta)\theta - \frac{f_3}{\lambda^l} = u(x)\mu(\xi), \quad \text{on} \quad \Omega.$$

Multiplying it by $(i\lambda + \xi^2 + \eta)^{-2}|\xi|$, we get

$$(i\lambda + \xi^2 + \eta)^{-2}|\xi| u(x)\mu(\xi) = (i\lambda + \xi^2 + \eta)^{-1}|\xi|\theta - (i\lambda + \xi^2 + \eta)^{-2}|\xi|\frac{f_3}{\lambda^l}, \ \forall x \in \Omega. \tag{42}$$

By taking absolute values of both sides of (42), integrating over $(-\infty, +\infty)$ with respect to the variable ξ and using Cauchy-Schwarz's inequality, we obtain

$$\mathcal{S}|u(x)| \leq \mathcal{U}\left(\int_{-\infty}^{+\infty} (\xi^2 + \eta)|\theta(x,\xi)|^2 \, d\xi\right)^{\frac{1}{2}} + \mathcal{V}\left(\int_{-\infty}^{+\infty} \left|\frac{f_3}{\lambda^l}\right|^2 d\xi\right)^{\frac{1}{2}} \tag{43}$$

with

$$\mathcal{S} = |\int_{-\infty}^{+\infty} (i\lambda + \xi^2 + \eta)^{-2}|\xi|\mu(\xi)\, d\xi| = \frac{|1-2\alpha|}{4}\frac{\pi}{|\sin\frac{(2\alpha+3)}{4}\pi|}|i\lambda + \eta|^{\frac{(2\alpha-5)}{4}},$$

$$\mathcal{U} = \left(\int_{-\infty}^{+\infty} |i\lambda + \xi^2 + \eta|^{-2}\, d\xi\right)^{\frac{1}{2}} \leq \sqrt{2}(\frac{\pi}{2})^{1/2}||\lambda| + \eta|^{-\frac{3}{4}},$$

$$\mathcal{V} = \left(\int_{-\infty}^{+\infty} (|i\lambda + \xi^2 + \eta|)^{-4}|\xi|^2\, d\xi\right)^{\frac{1}{2}} \leq 2\left(\frac{\pi}{16}||\lambda| + \eta|^{-\frac{5}{2}}\right)^{1/2}.$$

By using Young's inequality and integrating (43) over Ω, we obtain

$$\int_\Omega |u(x)|^2 dx \leq \frac{2\mathcal{U}^2}{\mathcal{S}^2}\int_\Omega\int_{-\infty}^{+\infty} (\xi^2+\eta)|\theta(x,\xi)|^2\, d\xi dx + \frac{2\mathcal{V}^2}{\mathcal{S}^2}\int_\Omega\int_{-\infty}^{+\infty}\left|\frac{f_3}{\lambda^l}\right|^2 d\xi dx.$$

It is easy to verify

$$\mathcal{S}^2 = O(|\lambda|^{\frac{2\alpha-5}{2}}), \quad \mathcal{V}^2 = O(|\lambda|^{-\frac{5}{2}}) \ \text{ and } \ \mathcal{U}^2 = O(|\lambda|^{-\frac{3}{2}}).$$

Using (38) and (39), we obtain

$$\int_\Omega |u(x)|^2 dx = \frac{o(1)}{\lambda^{\alpha-1+l}} + \frac{o(1)}{\lambda^{\alpha+2l}} = \frac{o(1)}{\lambda^{\alpha-1+l}}. \tag{44}$$

Lemma 4.9. *We have*

$$\int_\Omega |v(x)|^2 dx = \frac{o(1)}{\lambda^{1-\alpha}}.$$

Proof. Multiplying (38)$_1$ by $\bar{v}$ and (38)$_2$ by $\bar{u}$, summing and taking the real part, we get

$$\beta\|v\|^2 = \Re\left(\int_\Omega \frac{f_1\bar{v}}{\lambda^l} - \tilde{\gamma}\int_\Omega\int_{-\infty}^{+\infty}\theta\mu\bar{v} + \int_\Omega\frac{f_2\bar{u}}{\lambda^l}\right) + \beta\|u\|^2.$$

Using Cauchy-Schwarz's inequality and Young's inequality, we obtain for enough small ϵ

$$\beta\|v\|^2 \ \leq \ \frac{1}{2\lambda^l}\int_\Omega\left|\frac{f_1}{\sqrt{\epsilon}}\right|^2 dx + \frac{1}{2\lambda^l}\int_\Omega|\sqrt{\epsilon}\bar{v}|^2 dx + \frac{\tilde{\gamma}}{2}\int_\Omega\left|\int_{-\infty}^{+\infty}\frac{\theta\mu}{\sqrt{\epsilon}}d\xi\right|^2 dx$$

$$+ \quad \frac{\widetilde{\gamma}}{2}\int_{\Omega}|\bar{v}\sqrt{\epsilon}|^2dx + \frac{1}{2\lambda^l}\int_{\Omega}|f_2|^2 + \frac{1}{2}\|u\|^2 + \beta\|u\|^2.$$

From (39) and (3), we deduce that

$$\|v\|^2 = \frac{o(1)}{\lambda^l} + + \frac{o(1)}{\lambda^{\alpha-1+l}}.$$

Then

$$\begin{aligned}\|Y\|^2 &= \int_{\Omega}|u(x)|^2dx + \int_{\Omega}|v(x)|^2dx + \widetilde{\gamma}\int_{\Omega}\int_{-\infty}^{+\infty}|\theta(x,\xi)|^2\,d\xi dx \\ &= \frac{o(1)}{\lambda^{\alpha-1+l}}\end{aligned}$$

Taking $l = 1-\alpha$, we deduce that $\|Y\| = o(1)$ which is a contradiction with (36), consequently (33) holds. The proof is thus complete.

5. Lack of exponential stability.

In this section, we will study the lack of exponential decay of solution of the problem (12). We will use the following theorem.

Theorem 5.1 ([14]). *Assume that $\mathcal{A}$ is the generator of a strongly continuous semigroup of contractions $\{S(t)\}_{t\geq 0}$ on a Hilbert space X. Then $S(t)$ is exponentially stable if and only if*

$$\rho(\mathcal{A}) \supseteq \{i\lambda : \lambda \in \mathbb{R}\} \equiv i\mathbb{R} \tag{45}$$

and

$$\overline{\lim}_{|\lambda|\to\infty}\left\|(i\lambda I - \mathcal{A})^{-1}\right\|_{L(X)} < \infty. \tag{46}$$

Our main result is the following theorem.

Theorem 5.2. *The semigroup generated by the operator $\mathcal{A}$ is not exponentially stable.*

Proof. Let us consider the spectral problem

$$\begin{cases}\Delta\psi_n = -k_n\psi_n, & \text{in } \Omega,\\ \psi_n = 0, & \text{on } \partial\Omega,\end{cases} \tag{47}$$

where $\psi_n \in L^2(\Omega)$ and $(k_n), n \in N$ are positive and

$$k_n \longrightarrow +\infty \text{ as } n \longrightarrow \infty.$$

We will show that there exists a sequence of values λ_n such that

$$\left\|(i\lambda_n I - \mathcal{A})^{-1}\right\|_{L(X)} \longrightarrow \infty,$$

it is equivalent to prove that there exist a sequence of $F_n \in \mathcal{H}$ and a sequence of real numbers λ_n, such that

$$\left\|(i\lambda_n I - \mathcal{A})^{-1}F_n\right\|_{\mathcal{H}} \longrightarrow \infty,$$

where

$$i\lambda_n Y_n - \mathcal{A}Y_n = F_n, \tag{48}$$

with Y_n is not bounded.
To simplify the notation we will omit the index n.
The equation (48) becomes

$$\begin{cases} i\lambda u - i\Delta u + \widetilde{\gamma}\displaystyle\int_{-\infty}^{+\infty} \theta(x,\xi)\mu(\xi)d\xi + \beta v = f_1, \\ i\lambda v - i\Delta v - \beta u = f_2, \\ i\lambda\theta + (\xi^2+\eta)\theta - \mu(\xi)u = f_3. \end{cases} \tag{49}$$

Let us consider $f_1 = \psi_n,\ f_2 = k_n^2\psi_n$ and $f_3 = 0$, then the system (49) becomes

$$\begin{cases} i\lambda u - i\Delta u + \widetilde{\gamma}\displaystyle\int_{-\infty}^{+\infty} \frac{u}{i\lambda+\xi^2+\eta}\mu(\xi)^2 d\xi + \beta v = \psi_n, \\ i\lambda v - i\Delta v - \beta u = k_n^2\psi_n. \end{cases} \tag{50}$$

We look for solutions of the form

$$u = a\psi_n \text{ and } v = b\psi_n,$$

with $a,\ b \in \mathbb{C}$.
Then, using the expression of $\widetilde{\gamma}$ and lemma 2.3 the system (50) becomes

$$\begin{cases} 2iak_n + a\gamma(ik_n+\eta)^{\alpha-1} + \beta b = 1, \\ 2ibk_n - \alpha a = k_n^2, \end{cases} \tag{51}$$

therefore a and b satisfy the linear system

$$\begin{pmatrix} 2ik_n + \widetilde{\gamma}(ik_n+\eta)^{\alpha-1} & \beta \\ -\beta & 2ik_n \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ k_n^2 \end{pmatrix}.$$

Hence

$$b = \frac{2ik_n^3 + \widetilde{\gamma}k_n^2(ik_n+\eta)^{\alpha-1} + \beta}{-4k_n^2 + 2ik_n\widetilde{\gamma}(ik_n+\eta)^{\alpha-1} + \beta^2}.$$

Recalling that $v = b\psi_n$, then $\|v_n\|^2 \longrightarrow \infty$ as $n \longrightarrow +\infty$. Therefore, we get

$$\|Y_n\|^2 \longrightarrow +\infty \text{ as } n \longrightarrow \infty.$$

The proof is now complete.

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