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On options pricing under a quadratic stochastic process modulated GBM model

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Abstract

This paper deals with the European option-valuation problem under a prediction model for the underlying asset prices with an external impact. We suggest an alternative model for risky asset prices in which the parameters are dependent on a non-Markovian process. This generalizes the regime-switching models with continuous-time Markov-chain processes. A notable problem of this model is that the process embedded in the parameter of the stock price is non-Markovian; in addition, the market is incomplete. The change from the historical probability to a risk-neutral one is investigated, and the set of equivalent martingale measures is determined. In addition, an infinitesimal generator is obtained, which allows numerical simulations of the non-Markovian and the stock-price processes to be conducted. Several illustrations are provided.

Keywords: Quadratic stochastic process Brownian motion Non-Markov Options pricing Regime switching models .
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Financial contracts known as derivatives are a sophisticated instrument for hedging and managing investment risks. The values of these derivatives are dependent on changes in the value of the underlying financial assets (such as foreign currencies, government bonds, corporate equities, certificates of deposit, stock-price indices, or interest rates) or a commodity (such as gold, petroleum, copper, wheat, coffee, or cattle) [16]. Over the past few decades, many researchers have examined the application of derivatives in hedging against risk exposure, determining underlying asset prices, increasing the efficiency of markets, and accessing markets or assets that are rarely available.

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Options, futures, forwards, and swaps are some examples of financial derivatives, and the modern financial markets are unimaginable without them. An option is a contract between two traders that grants one of them the right (but not the obligation) to purchase or sell an asset at a predetermined price. American, European, and exotic options are some examples of these derivatives [6].

Financial economics has long been devoted to the study of pricing options, and this is an intensely studied problem. Since the pioneering work of Black and Scholes [1] and Merton [18], there has been a plethora of articles on both the theory and practice of option pricing. In the Nobel-prize-winning contribution of the Black–Scholes–Merton model, the problem was solved by assuming that the original asset follows geometric Brownian motion (GBM). Subsequently, a variety of different option-valuation models have been proposed and tested to overcome several limitations of this model. One of these stems from the notion that the price of an option is not affected by external events that impact the price of the underlying asset. In a financial market, it is universally accepted that the underlying asset price can be subject to exogenous factors. For example, the recent COVID-19 pandemic has certainly had an impact on the prices of financial assets. Models with regime-switching were developed to overcome this fault in the Black–Scholes–Merton model (see, for instance, [2, 7]).

Studying the European option-valuation problem under a prediction model for the underlying asset prices with an external impact is an active and open research area. The different parameters of the model are assumed to be contingent on other circumstances not detected from the market. A common technique to model these circumstances is to use continuous-time Markov processes [8]. While research on Markov processes and their applications has a long tradition, not all physical quantities can be modeled by Markov processes. Most random processes are non-Markovian. These include polymers with extended volume [5], first-passage problems in three dimensions [10], the Kramers problem in diffusion [12, 17], and population genetics models [9]. Non-Markovian processes have manifold applications in many fields, including robotics [11], thermodynamics [22], and atomic physics [15].

Markovian semigroups have been used to provide effective descriptions of the evolution of several important systems. However, the recent development of quantum systems has led to the proposal of a more refined approach that takes into account non-Markovian memory effects, which are completely neglected in the Markovian regime. Hence, non-Markovian quantum dynamics has attracted a lot of attention in recent years, and there is already a vast body of literature dealing both with several theoretical approaches and various experimental realizations [3, 21].

Most of the existing results related to financial derivatives are based on Markov systems obeying the semigroup property. Therefore, it is natural to start considering processes that do not follow the semigroup property. In this paper, we employ a non-Markovian-process-modulated GBM model for the underlying asset model. The non-Markovian process is built from two continuous-time Markov chains (CTMCs). It is non-Markovian mainly because it does not satisfy the semigroup property. The model considered is a generalization of CTMC regime-switching models (see, for instance, [8, 2]).

The remainder of this paper is organized as follows. In Section 1, the non-Markovian-modulated GBM model is presented. The change of probability is discussed in Section 2. Section 3 is devoted to numerical simulations, and we conclude in Section 4.

1. Non-Markovian-modulated GBM model

Consider a standard Brownian motion $(W_t)_{t\in[0,T]}$, and let $(\Omega, \mathcal{F}, P)$ be a probability space in which there is a filtration $(\mathcal{F}_t)_{t\in[0,T]}$ with $\mathcal{F} := \mathcal{F}_T$. As in [8], the states of the economy are modeled by a continuous process $(X_t)_{t\in[0,T]}$ defined by its finite state space $\mathcal{S} = \{s_1, s_2, \dots, s_N\}$, and this is assumed to be identified to a finite set of unit vectors $\{e_1, e_2, \dots, e_N\}$, where $e_i = (0, \dots 1, \dots 0) \in R^N$. Thus, the model treated in this paper suggests having two sources of randomness: W and X. We assume that X_t and W_t are independent.

Contrary to similar research works that make use of Markov processes, our model uses a non-Markovian process. The process X considered here is assumed to be obtained from the combination of two CTMC processes $(Y_t)_{t\in[0,T])}$ and $(Z_t)_{t\in[0,T])}$. More precisely, X has a transition probability matrix defined by

$$\Theta_t = \theta P_t, \tag{1}$$

where P_t is the transition probability of the continuous-time Markov process Y_t, and $\theta = \theta_{t=1}$, wherein θ_t is the transition probability of the Markov process Z_t. We notice that the dynamical system Θ_t is divisible one, i.e., $\Theta_t = V_{t,s}\Theta_s$, where $V_{t,s} = P_{t-s}$, for any $t \geq s$ (see [3]). Such types of process have been investigated previously [23, 9]. However, we stress that Θ_t does not satisfy the semigroup property. Therefore, we refer this process as non-Markovian, while in physical literature, this kind of system is called Markovian [3].

In this work, there are two assets in the markets, and time is moving from $t = 0$ to $t = T$. The first asset A with a price $(A_t)_{t\in[0,T]}$ is generated by the following process:

$$dA_t = r_t A_t dt, \quad t \in [0, T], \tag{2}$$

where $r_t > 0$ is the interest rate. The second asset is an underlying risky asset denoted by $(S_t)_{t\in[0,T]}$ driven by the stochastic differential equation

$$dS_t = \mu_t S_t dt + \sigma_t S_t dW_t, \quad S_0 \geq 0, \tag{3}$$

where μ and σ are the expected return and volatility, respectively. Moreover, we have

$$\begin{aligned} r_t :&= \langle r, X_t\rangle = \sum_{k=1}^{N} r_k 1_{\{X_t=s_k\}}, \quad \text{where } r := (r_1, \ldots, r_N), \\ \mu_t :&= \langle \mu, X_t\rangle = \sum_{k=1}^{N} \mu_k 1_{\{X_t=s_k\}}, \quad \text{where } \mu := (\mu_1, \ldots, \mu_N), \\ \sigma_t :&= \langle \sigma, X_t\rangle = \sum_{k=1}^{N} \sigma_k 1_{\{X_t=s_k\}}, \quad \text{where } \sigma := (\sigma_1, \ldots, \sigma_N). \end{aligned}$$

The aim is to study European call options under the above model. Consider a European call option built on stock S with price S_t at time $t \in [0, T]$ with expression under the risk-neutral probability Q given by (7). The strike price is denoted by K, and T is the expiration date. The payoff of the European call option is then given by the function $h(S_T) = (S_T - K)^+ := \max(S_T - K, 0)$.

2. Change of probability

The pricing of options imposes general working under a probability, ensuring the completeness of the market and that there are no arbitrage opportunities. In other words, there is no profit without taking risks, and each contingent claim is attainable. To guarantee the non-arbitrage opportunity condition, the fundamental theorem of asset pricing links the absence of arbitrage opportunities to the existence of a probability equivalent to the historical probability, denoted P in our settings, under which the discounted stock-price process is a martingale. The equivalent probability, if it exists, is called an equivalent martingale measure (EMM), and we will be using a P-EMM to classify the P EMMs. The second fundamental theorem states that the market is complete if there is one and only one EMM, and it is incomplete if there are many EMMs. The details of these results are described in [13, 14, 4]. If Q is a P-EMM, then it can be characterized by its Radon–Nikodym derivative, denoted by $\rho_T := \frac{dQ}{dP}$, and we have:

$$Q(A) = E^P\left[\rho_T 1_A\right], \quad A \in \mathcal{P}(\Omega),$$

where ρ_T is strictly positive P-a.e., and $E^P[\rho_T] = E_P[\rho_T 1_\Omega] = Q(\Omega) = 1$. Following [8], the logarithmic return of the underlying asset is $(L_t)_{t\in[0,T]}$, and using the Itô formula applied to the stochastic differential equation (3), L_t can be expressed as

$$L_t = \int_0^t \left(\mu_s - \frac{\sigma_s^2}{2}\right) ds + \int_0^t \sigma_s dW_s. \tag{4}$$

Let $\mathcal{F}_t^L$ be the filtration generated by L_t, and the filtration generated by X_t is $\mathcal{F}_t^X := \sigma(X)$; then, we assume that the global filtration is $\mathcal{H}_t := \mathcal{F}_t^L \vee \mathcal{F}_t^X$. Proposition 2.1 follows [8] to determine the expression of the Radon–Nikodym derivative ρ_T of a given P-EMM Q.

Proposition 2.1. *The Radon–Nikodym derivative of the P-EMM Q is given by*

$$\rho_T \mid_{\mathcal{F}_t^X} = \exp\left(\int_0^T \alpha_t dW_t - \frac{1}{2}\int_0^T \alpha_t^2 dt\right), \tag{5}$$

where $(\alpha_t)_{t\in[0,T]}$ is defined by $\alpha_t := \langle \alpha, X_t\rangle = \sum_{k=1}^N \alpha_k 1_{\{X_t=s_k\}}$, and where for $k \in \{1,\ldots,N\}$,

$$\alpha_k := \frac{r_k - \mu_k}{\sigma_k}. \tag{6}$$

The stock price can be written under Q as

$$dS_t = r_t S_t + \sigma_t S_t dW_t^Q, \quad t \in [0,T], \tag{7}$$

where the process $W^Q := (W_t^Q)_{t\in[0,T]}$ is the Q-Brownian motion defined by the Girsanov theorem as $W_t^Q := W_t - \alpha_t$.

Proof. The proof of Proposition 2.1 is based on [8]. First, let $\beta^Q := (\beta_t^Q)_{t\in[0,T]}$ be the parameter of the regime-switching Esscher transform defined by

$$\beta_t := \langle \beta_t, X_t\rangle = \sum_{k=1}^N \beta_k 1_{\{X_t=s_k\}}, \quad \text{where } \beta := (\beta_1,\ldots,\beta_N).$$

As in [8], the Radon–Nikodym derivative of the regime-switching Esscher transform can be written as

$$\rho_t \mid_{\mathcal{F}_t^X} = \left(\int_0^t \beta_s \sigma_s dW_s - \frac{1}{2}\int_0^t \beta_s^2 \sigma_s^2 ds\right). \tag{8}$$

Moreover, the discounted-price martingale condition under the P-EMM Q is affected by uncertainty caused by the non-Markovian process X; therefore, it is expressed as

$$\begin{aligned} S_0 &= E^Q\left[e^{-\int_0^t r_s ds} S_t \mid \mathcal{F}_t^X\right] \\ &= S_0 E^Q\left[\exp\left(-\int_0^t r_s ds + L_t\right) \mid \mathcal{F}_t^X\right] \\ &= S_0 E^Q\left[\exp\left(\int_0^t \left(-r_s + \mu_s - \frac{\sigma_s^2}{2}\right) ds + \int_0^t \sigma_s dW_s\right) \mid \mathcal{F}_t^X\right] \\ &= S_0 E^Q\left[\exp\left(\int_0^t \left(-r_s + \mu_s - \frac{\sigma_s^2}{2}\right) ds + \int_0^t \sigma_s dW_s\right) \mid \mathcal{F}_t^X\right], \end{aligned} \tag{9}$$

where we have used $S_t = S_0 e^{L_t}$ (Eq. (4)). Notice that

$$\exp\left(\int_0^t \left(-r_s + \mu_s - \frac{\sigma_s^2}{2}(1+\beta_s^2)\right) ds + \int_0^t \beta_s(1+\sigma_s) dW_s\right) \mid_{\mathcal{F}_t^X}$$

follows a normal distribution:

$$N\left(\int_0^t (-r_s + \mu_s - \frac{\sigma_s}{2}(1+\beta_s^2))ds, \int_0^t \sigma_s^2(1+\beta_s)^2)ds\right).$$

Therefore, using Bayes' rule together with (8) to move to the probability P, the martingale condition becomes

$$\begin{aligned} 1 &= E^P\left[\exp\left(\int_0^t\left(-r_s+\mu_s-\frac{\sigma_s^2}{2}(1+\beta_s^2)\right)ds+\int_0^t\sigma_s(1+\beta_s)dW_s\right)\mid\mathcal{F}_t^X\right] \\ &= \exp\left[\int_0^t(-r_s+\mu_s-\frac{\sigma_s^2}{2}(1+\beta_s^2))ds+\frac{1}{2}\int_0^t\sigma_s^2(1+\beta_s)^2)ds\right]. \end{aligned}$$

Thus,

$$\int_0^t(-r_s+\mu_s-\frac{\sigma_s^2}{2}(1+\beta_s^2))ds+\frac{1}{2}\int_0^t\sigma_s^2(1+\beta_s)^2)ds=0,$$

and

$$-r_t+\mu_t+\sigma_t^2\beta_t=0.$$

Then,

$$\beta_t=\frac{\mu_t-r_t}{\sigma_t^2}\quad\text{and}\quad\alpha_t=\beta_t\sigma_t=\frac{\mu_t-r_t}{\sigma_t}. \tag{10}$$

This leads to Eqs. (5)–(6). Eq. (7) can be obtained using the Girsanov theorem and the relation between W and W^Q. The proof is completed. □

The above results can be used to find the EMM that minimizes the entropy, called the MEMM. In fact, as in [8], it can be shown that Eq. (6) determines the MEMM.

3. Numerical simulations

Simulations of a CTMC can in general be conducted on the basis of its infinitesimal generator. The process X considered in this paper is a non-Markovian process. Thus, the regular simulation methodology for CTMC cannot be applied. Nonetheless, it is natural to think about getting sample paths for X by investigating an "infinitesimal generator" for it. This is discussed in the next subsection.

3.1. Infinitesimal generator for X

The process X is built on the CTMC Y with transition matrix $P_t = P(t)$ and the CTMC Z with transition matrix θ_t. We then define X to be the process with a transition matrix defined by Eq. (1), where $\theta=\theta_{t=1}$. The $N\times N$ matrix $P(t)$ is given by

$$P(t)=\begin{pmatrix} P_{1,1}(t) & P_{1,2}(t) & \dots & P_{1,N-1}(t) & P_{1,N}(t)\\ P_{2,1}(t) & P_{2,2}(t) & \dots & P_{2,N-1}(t) & P_{2,N}(t)\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ P_{N,1}(t) & P_{N,2}(t) & \dots & P_{N,N-1}(t) & P_{N,N}(t)\end{pmatrix},$$

where the transition probabilities $P_{ij}(t)=P\{Y_{t+s}=j\mid Y_s=i\}$ for all $s\geq 0$ are independent of s, and $P_{ij}(t)\geq 0, \Sigma_{j=1}^N P_{ij}=1$ for each $i\in\{1,\dots,N\}$. The relation between the transition matrix of Y, P_t, and its infinitesimal generator, which is an $N\times N$ matrix represented here by A, the semigroup property

$$P_{t+h}=P_tP_h=P_hP_t \tag{11}$$

is used to get

$$P(t)=P(0)e^{At}=e^{At},$$

where $A=P_0'$, $a_{ij}(t)\geq 0$ for $j\neq i$, and $a_{ii}(t)=-\Sigma_{j\neq i}a_{ij}(t)$ for $t\geq 0$.

Remark 3.1. *For the CTMC Y, if we define τ_i as the holding time, or the time spent in state s_i, then τ_i is an exponential random variable with parameter $a_{ii}(t)=-\Sigma_{j\neq i}a_{ij}(t)$.*

For more details about CTMC processes and Markov chains, the reader can consult two published books [19, 20]. Proposition 3.2 deals with the infinitesimal generator for our non-Markovian process X.

Proposition 3.2. *We assume that the infinitesimal generator A of CTMC Y and the transition matrix of CTMC Z at $t = 1$ $(\theta_{t=1})$ commute. Then, Θ_t, the transition matrix of process X given by Eq. (1), can be written as*

$$\Theta_t = \theta e^{At}. \tag{12}$$

Proof. Let Θ_t' be the derivative of Θ_t. Then,

$$\begin{aligned}
\Theta_t' &= \lim_{h\to 0+} \frac{\Theta_{t+h} - \Theta_t}{h} \\
&= \lim_{h\to 0+} \frac{(\theta P)_{t+h} - (\theta P)_t}{h} \\
&= \lim_{h\to 0+} \frac{\theta P_{t+h} - \theta P_t}{h} \\
&= \theta \lim_{h\to 0+} \frac{P_{t+h} - P_t}{h} \\
&= \theta P_t' = \theta A P_t = A\theta P_t = A\Theta_t.
\end{aligned}$$

Then, $\Theta_t = \theta e^{At}$, since $\Theta_0 = \theta P_0 = \theta$, where we have used the fact that $P_0 = I$ (the identity matrix). □

Remark 3.3. *Proposition 12 tells us that if we impose $\theta = Id$, then process X is exactly CTMC Y. This will be useful in the simulation of sample paths for the non-Markovian process X, especially to find the probability distribution of the holding time at a given state s_i, for $i \in \{1, \ldots, N\}$.*

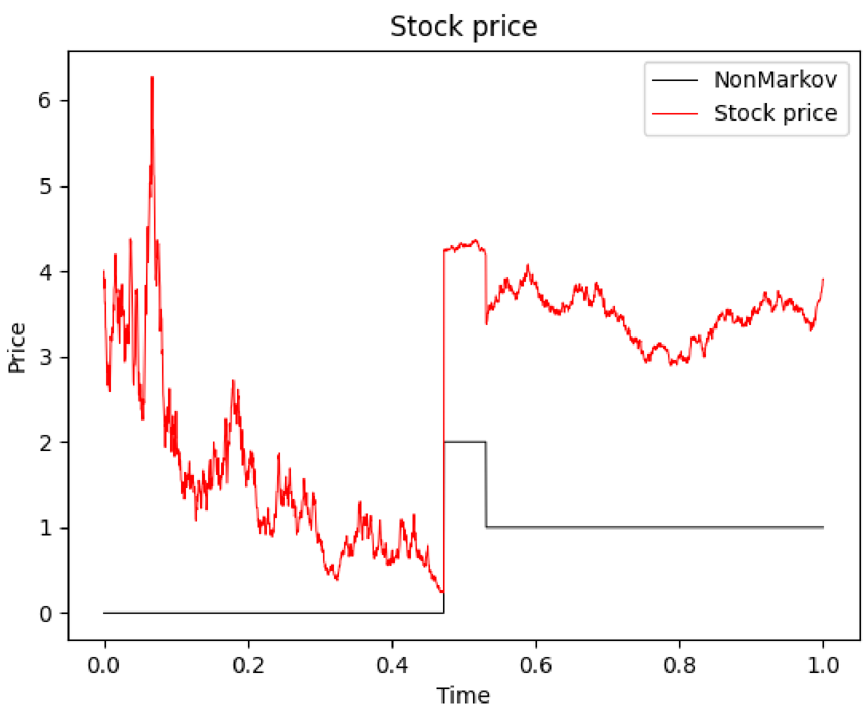

Figure 1: Realizations of the stock price with the non-Markovian process. First run of the simulations.

3.2. Simulations of the non-Markovian process X_t and the stock price

The stock price is given by

$$S_t = S_0 \exp\left[\int_0^t \left(r_s - \frac{\sigma_s^2}{2}\right) ds + \int_0^t \sigma_s dW_s^Q\right]. \tag{13}$$

First, we discretize the time into M time steps t_k of equal size $\Delta t = t_{k+1} - t_k = \frac{T}{M}$, for $k = 0, \ldots, M-1$.

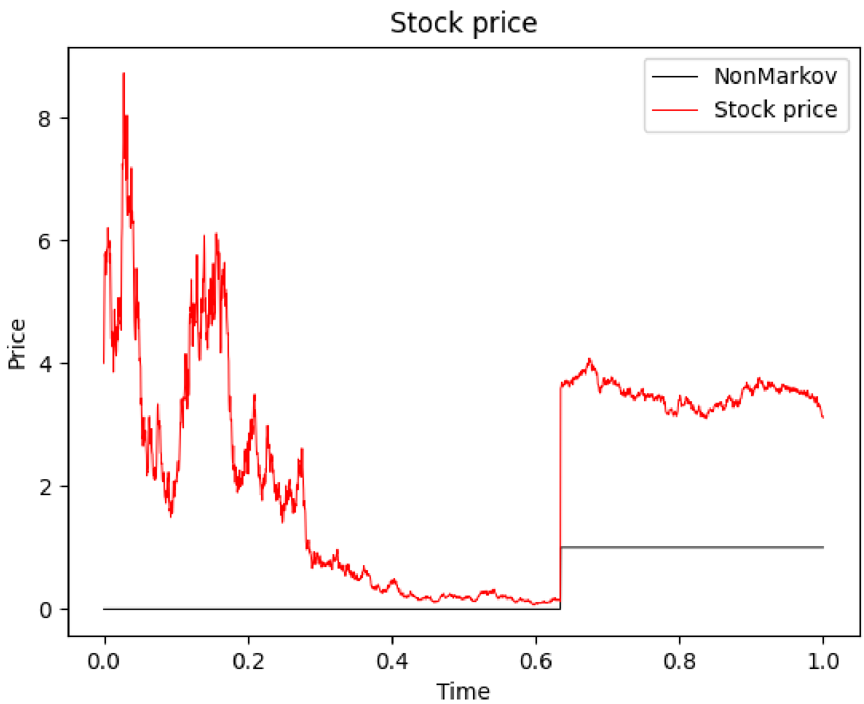

Figure 2: Realizations of the stock price with the non-Markovian process.

A sample trajectory for the stock price $(S_k)_{k\in\{0,1,\ldots,M\}}$ at maturity T can be obtained using the following algorithm:

1. We simulate a trajectory for the Brownian motion: $(W_k^Q)_{k=0,\ldots,M-1}$ as $\Delta W_k^Q : W_{k+1}^Q - W_k^Q$, a normally distributed random variable $N(0, \Delta t)$.
2. We independently simulate a trajectory for the non-Markovian process $(X_k)_{k=0,\ldots,M-1}$ using the definition of its transition matrix Θ_t in (1) and its infinitesimal generator from Proposition 3.2.
3. For $k = 1$, all the parameters of the model are known from the previous steps, and we use (13) to get S_1.
4. Repeat the previous step to $k = M - 1$.

This algorithm was coded in Python, where the process X_t was assumed to have three possible states, $X_t \in \{0, 1, 2\}$, and we chose $S_0 = 4$, $T = 1$, and $M = 2000$.

State one, when $X_t = 0$, corresponds to a bad economic situation for the stock price. The interest-rate value is $r(0) = 0.01$, and we take a high value for the volatility $\sigma(0) = 3$. When the economy is in normal situations, $X_t = 1$, and the values are interest rate $r(1) = 0.05$ and volatility $\sigma(1) = 0.4$. The last state, state three, when $X_t = 2$, represents good economic conditions; in this case, the interest rate is $r(2) = 0.2$ and the volatility is $\sigma(2) = 0.15$.

Figs. 2–3 present illustrations of three realizations of the stock prices under the non-Markovian-modulated GBM model. They concord with the reality mainly when an external event occurs and has an impact on the stock price.

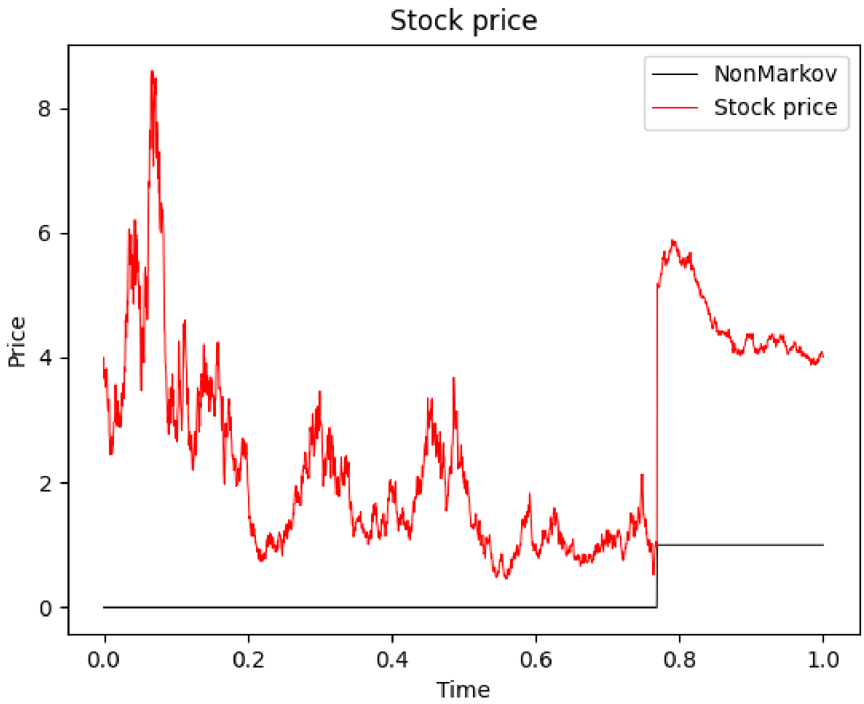

Figure 3: Realizations of the stock price with the non-Markovian process. Second run of the simulations.

4. Conclusion

Regime-switching models for an underlying asset price are usually built using Markov processes. In this work, we developed an alternative model in which the impact of an external event is represented by a non-Markovian process. The stock price follows non-Markovian-modulated GBM. After setting up the model, to ensure the non-arbitrage opportunity condition, the martingale condition for the discounted price was employed to determine the EMM. Moreover, an infinitesimal generator for the non-Markovian process used in the model was obtained. Numerical methods were then applied, and simulations for the non-Markovian and stock-price processes were conducted. Several illustrations were given, and these show the appropriateness of the model. The studied model can be seen as a generalization of the regime-switching CTMC-modulated GBM model.

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