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An open discussion: Interpolative Metric Spaces

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Abstract

The main goal of this paper is to introduce a new abstract structure (so called, interpolative metric space) as a generalization of a standard metric space. We shall consider the analog of Banach Mapping Principle in the context of this new structure.

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1. Introduction

Stefan Banach made the metric fixed point theory a separate research topic more than a century ago. In other words, the metric fixed theory has successfully survived a century with fruitful results and remarkable applications for almost all quantitative sciences. This adventure, which started with Banach's demonstration that every contraction mapping has only one fixed point in the complete norm space, has reached a completely different (higher) point today. After the pioneer fixed point result of Banach, a serious number of researchers have paid attention to this research field and produced significant results.

It would not be wrong if we say that the fixed point theory is divided into a few main streams. One of the trends in improving, generalizing, and advancing the metric fixed point theory is to consider new type contractions and check whether a mapping satisfying the newly defined contraction provides a unique fixed point. Among all, the Kannan type, the Meir-Keeler type, the Reich type, the Ćirić type, the Rus type, and the Hardy-Rogers type contractions are paragons, see e.g. [1, 2].

Another trend is proposing a new abstract structure. After examining the consistency of the new structure, researchers investigate the existence and uniqueness of the fixed point of certain operators. As examples, we can count, quasi metric spaces, ultra-metric spaces, b-metric spaces, partial metric spaces, dislocated metric spaces, and so on [1, 2].

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The aim of this paper is to follow the second trend by defining the notion of interpolative metric.
In what follows, we shall state the definition of (α, c)-interpolative metric.

Definition 1.1. *Let X be a nonempty set. We say that $d : X \times X \to [0, +\infty)$ is (α, c)-interpolative metric if*

(m1) $d(x, y) = 0$, *if and only if,* $x = y$ *for all* $x, y \in X$

(m2) $d(x, y) = d(y, x)$, *for all* $x, y \in X$,

(m3) *there exist an* $\alpha \in (0, 1)$ *and* $c \geq 0$ *such that*

$$d(x,y) \leq d(x,z) + d(z,y) + c\left[(d(x,z))^{\alpha}(d(z,y))^{1-\alpha}\right],$$

for all $(x, y, z) \in X \times X \times X$.

Then, we call (X, d) an (α, c)-interpolative metric space.

Note that the notion of interpolative metric as a natural generalization of a standard metric space. In other words, each metric space can be considered an (α, c)-interpolative metric space with $c = 0$. On the other hand, the converse of the statement is not true. The following example illustrate why the converse is invalid.

Example 1.1. *First consider a standard metric space (X, δ). We construct a distance function $d : X \times X \to [0, \infty)$ in the following way*

$$d(x, y) := \delta(x, y)(\delta(x, y) + 1).$$

Employing the fact that δ is a metric on X, the conditions $(m1)$ and $(m2)$ are observed trivially. For $(m3)$, it is sufficient to set $c \geq 2$ for any $\alpha \in (0, 1)$. Consequently, we derive that (X, d) is $(\frac{1}{2}, 2)$-interpolative metric space.

In fact, we derive it easily, as follows;

$$\begin{aligned}
d(x,y) &= \delta(x,y)(\delta(x,y)+1) \\
&\leq (\delta(x,z)+\delta(z,y))(\delta(x,z)+\delta(z,y)+1) \\
&\leq (\delta(x,z)+\delta(z,y))(\delta(x,z)+\delta(z,y)+1) \\
&\leq [\delta(x,z)(\delta(x,z)+1)+\delta(x,z)\delta(z,y)] + [\delta(z,y)(\delta(z,y)+1)+\delta(z,y)\delta(x,z)] \\
&\leq [\delta(x,z)(\delta(x,z)+1)] + [\delta(z,y)(\delta(z,y)+1)] + 2\delta(x,z)\delta(z,y) \\
&\leq d(x,z)+d(z,y)+2\,(\delta(x,z))^{\frac{1}{2}}\,(\delta(x,z))^{\frac{1}{2}}\,(\delta(z,y))^{\frac{1}{2}}\,(\delta(z,y))^{\frac{1}{2}} \\
&\leq d(x,z)+d(z,y)+2\,(\delta(x,z))^{\frac{1}{2}}\,[\delta(x,z)+1]^{\frac{1}{2}}\,(\delta(z,y))^{\frac{1}{2}}\,[\delta(z,y)+1]^{\frac{1}{2}} \\
&\leq d(x,z)+d(z,y)+2(d(x,z))^{\frac{1}{2}}(d(z,y))^{\frac{1}{2}}
\end{aligned}$$

It is obvious that this distance function $d(x, y)$ does not form metric.

Example 1.2. *In this example, to illustrate the idea, we can consider the standard Euclid metric, as a special case of the above example. Let $(X = [0, \infty, \delta := |x-y|)$. We construct a distance function $d : X \times X \to [0, \infty)$ in the following way*

$$d(x, y) := |x - y|(|x - y| + A) = |x - y|^2 + A|x - y|, \text{ for all } x \in X,$$

where $A > 0$.

We skip the proof since it is obtain by verbatim of the proof in the above example.

Suppose that $r > 0$ and $x \in X$. Denote

$$\mathfrak{B}(x, r) = \{y \in X : d(x, y) < r\}$$

as an open ball in (α, c)-interpolative metric space (X, d)

Definition 1.2. *Let (X, d) be a (α, c)-interpolative metric space and let $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ converges to x in X, if and only if, $d(x_n, x) \to 0$, as $n \to \infty$.*

Definition 1.3. *Let (X, d) be a (α, c)-interpolative metric space and let $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is a Cauchy sequence in X, if and only if, $\lim_{n\to\infty} \sup\{d(x_n, x_m) : m > n\} = 0$.*

Definition 1.4. *Let (X, d) be a (α, c)-interpolative metric space. We say that (X, d) is a complete (α, c)-interpolative metric space if every Cauchy sequence converges in X.*

In this paper, we introduce the notion of (α, c)-interpolative metric to open a discussion on the novelty of the generalization. In addition, we shall obtain the analog of the Banach Contraction Principle in this new setting to open a new framework for the metric fixed point theory.

2. Main Result

In this section, we shall derive the analog of the renowned Banach's fixed point theorem in the context of an (α, c)-interpolative metric spaces. First, we state analog of the Banach Contraction Principle in the setting of an (α, c)-interpolative metric spaces.

Theorem 2.1. *Let (X, d) be a (α, c)-interpolative metric space and let $T : X \to X$ be a mapping. Suppose that there exists q with $0 < q < 1$ such that*

$$d(Tx, Ty) \leq qd(x, y) \tag{1}$$

for all $x, y \in X$. Then, T possesses a unique fixed point in X.

Proof. First of all we shall construct an iterative sequence starting with an arbitrary point in X: Let $x \in X$ and redefine it as $x_0 := x$. Based on the Picard idea, we set $Tx_0 := x_1$, and $Tx_1 := x_2$, and inductively we derive a sequence $\{x_n\}$ in a way that $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}_0$. Note that the case $x_{n_0} = x_{n_0+1}$ for any $n_0 \in \mathbb{N}_0$ finished the proof since $x_{n_0} = x_{n_0+1} = Tx_{n_0}$; that is, x_{n_0} forms the required fixed point which terminates the proof. Keeping this discussion in mind, without loss of generality, we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}_0$, throughout the proof. In other words, the image of the successive terms in $\{x_n\}$ under the distance function d is strictly positive, that is, $d(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N}_0$.

On account of the assumption (1) of the theorem, we find that

$$d(x_n, x_{n+1}) \quad \leq \quad qd(x_n, x_{n-1}), \text{ for all } n \in \mathbb{N}_0. \tag{2}$$

Regarding the above inequality, by iteration, we find that

$$d(x_n, x_{n+1}) \leq q^n d(x_0, x_1), \text{ for all } n \in \mathbb{N}_0 \tag{3}$$

Setting $n \to \infty$ in the limit 2 yields that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{4}$$

Furthermore, the observed limit (4) implies also that there exists $k \in \mathbb{N}$ such that

$$d(x_n, x_{n+1}) \leq 1 \text{ for all } n \geq k. \tag{5}$$

In what follows, we indicate that the obtained sequence has no period point: To prove this claim, we suppose that, $m, n \in \mathbb{N}$ and $m > n > k$. If $x_n = x_m$, we have $T^m(x_0) = T^n(x_0)$. So, we find that $T^{m-n}(T^n(x_0)) = T^n(x_0)$. Hence, we conclude $T^n(x_0)$ is the fixed point of T^{m-n}. Also,

$$T(T^{m-n}(T^n(x_0))) = T^{m-n}(T(T^n(x_0))) = T(T^n(x_0))$$

On other words, $T(T^n(x_0))$ is the fixed point of T^{m-n}. Consequently, we find that $T(T^n(x_0)) = T^n(x_0)$. As a result, $T^n(x_0)$ forms the desired fixed point of T. In conclusion, we can suppose that $x_n \neq x_m$, without loss of generality.

As a next step, we shall demonstrate that the constructed recursive sequence $\{x_n\}$ fulfils Cauchy criteria. We shall prove our assertion by standard induction. As a first, we observe the following limit:

$$d(x_n, x_{n+2}) \quad \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + c\left[(d(x_n, x_{n+1}))^{\alpha}\,(d(x_{n+1}, x_{n+2}))^{1-\alpha}\right] \tag{6}$$

By taking $n \to \infty$ and regarding (4), we find that

$$\lim_{n\to\infty} d(x_n, x_{n+2}) = 0. \tag{7}$$

In addition, we find that

$$d(x_n, x_{n+3}) \quad \leq d(x_n, x_{n+2}) + d(x_{n+2}, x_{n+3}) + c\left[(d(x_n, x_{n+2}))^{\alpha}\,(d(x_{n+1}, x_{n+3}))^{1-\alpha}\right] \tag{8}$$

By combining the limit (4) and inequality (8), we deduce that

$$\lim_{n\to\infty} d(x_n, x_{n+3}) = 0. \tag{9}$$

Regarding the induction steps, we shall presume the following limit

$$\lim_{n\to\infty} d(x_n, x_{n+r}) = 0, \text{ for some } r \in \mathbb{N}. \tag{10}$$

Due to the statement of the given theorem, we find

$$d(x_n, x_{n+r+1}) \quad \leq d(x_n, x_{n+r}) + d(x_{n+r}, x_{n+r+1}) + c\left[(d(x_n, x_{n+r}))^{\alpha}\,(d(x_{n+r}, x_{n+r+1}))^{1-\alpha}\right] \tag{11}$$

By employing the limits (4) and (10), as $n \to \infty$, we obtain that

$$\lim_{n\to\infty} d(x_n, x_{n+r+1}) = 0. \tag{12}$$

Consequently, we deduce that the recursively constructed sequence $\{x_n\}$ forms Cauchy. Regarding that (X, d) is a complete (α, c)-interpolative metric space, the sequence $\{x_n\}$ converges to $z \in X$. We assert that z is the fixed point of T. On the contrary, assume $d(z, Tz) > 0$. Note that

$$d(x_{n+1}, Tz) = d(Tx_n, Tz) \leq q d(x_n, z). \tag{13}$$

Finally, by letting $n \to \infty$ from both sides of 13, we conclude that $q < 1$, a contradiction. Hence, $Tz = z$ forms the fixed point of T in X. Notice also that the uniqueness of the fixed point is clear from (1). □

3. Conclusion

This paper proposes a new structure, new abstract space to deal with the problems of metric fixed point theory. Regarding the techniques of interpolation, it may help to get better estimation in calculation. The main purpose of this paper is to open a discussion in getting possible better results in the context of the metric fixed point theory.

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