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Reconstruct an unknown source term on the right hand side of the time-fractional diffusion equation with Caputo-Hadamard derivative

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Abstract

This study considers an inverse source problem of a time-space fractional diffusion equation. In general this inverse problem is ill-posed in the sense of Hadamard. We provide a new non-stationary iterated quasi-boundary value regularization technique for reconstructing the source function. By choosing the regularization parameters (a priori; posterior decision rules), we determine the convergence rates.

Keywords: Caputo-Hadamard derivative, Inverse source problem, parabolic equation, memory term, regularization method.
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1. Introduction

Fractional models have garnered significant interest from numerous researchers over the past three decades due to their effective applications in diverse fields [1, 2]. Several areas can be mentioned as anomalous diffusion, anomalous transport, mechanics in porous media, viscoelastic mechanics, option pricing, image processing, and signal processing [3–7]. In this paper, we study the following problem with Caputo-Hadamard fractional derivative as follows:

$$\begin{cases} {}_{CH}\mathbb{D}^{\gamma}_{b,t}\theta(x,t) + (-\Delta)^{\sigma}\theta(x,t) = \mathcal{F}(x)\mathcal{Q}(t), & x \in \mathbb{X}, t \in (b,\mathcal{T}], 0 < \gamma < 1,\ b > 1, \\ \theta(x,b) = a(x), & x \in \mathbb{X}, \\ \theta(x,t) = 0, & x \in \partial\mathbb{X}, t \in (b,\mathcal{T}], \\ \theta(x,\mathcal{T}) = b(x), & x \in \mathbb{X}, \end{cases} \tag{1}$$

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where $b(x)$ is the terminal data, where $0 < \mathcal{Q}_1 \leq \mathcal{Q} \leq \mathcal{Q}_2$, and ${}_{CH}\mathbb{D}^{\gamma}_{b,t}$ is the Caputo-Hadamard fractional derivative of the order γ defined by

$$ {}_{CH}\mathcal{D}^{\gamma}_{c,t}\theta(x,t) = \frac{1}{\Gamma(1-\gamma)}\int_b^t \big(\log\frac{t}{z}\big)^{-\gamma}\Upsilon\theta(x,z)dz, 0<\gamma<1,$$

where $\Upsilon = t\dfrac{d}{dt}, \Gamma(x)$ is a Gamma function. In problem (1), the source term $f(x)$ is unknown, and the initial value $a(x)$ is known. The final value $\theta(x,\mathcal{T}) = b(x)$ is the given data. This paper uses the final value $b(x)$ to identify the source term $f(x)$. Assume that the couple exact data $\{a, b\}$ and the measurement data $\{a^\delta, b^\delta\}$ satisfy

$$\|a^\delta - a\|_{L^2(\mathbb{X})} + \|b^\delta - b\|_{L^2(\mathbb{X})} \leq \delta.$$

where $\|\cdot\|$ is the $L^2(\mathbb{X})$ norm and $\delta > 0$ is the measurement error.

The investigation of the problem of restoring the source function also receives a lot of attention from mathematicians around the world. Some related works can be mentioned as follows: Yang and Fu [8] adopted the quasi-reversibility regularization method, Sakamoto and Yamamoto [9] studied the regularity of the general initial boundary value problems and presented several results related to the inverse problems, and Fujishiro and Kian [10] recovered the source term from an interior measurement in fractional diffusion equations. Ting Wei used the Modified Quasi Boundary value method [11, 12] to deal with the inverse source problem and obtain two kinds of convergence rates by using an a priori and an a posteriori regularization parameter choice rule, respectively. Fang Yang [13] with the Simplified Tikhonov, see in [14], and the truncation method, see [15].

In this work, we propose a nonstationary iterated QBVM (NIQBVM), see in [16] for reconstructing the space-dependent source $f(x)$ in (1). The key idea of this method is to modify the terminal data into (1) by

$$\theta^k(x,\mathcal{T}) + \alpha_k\big(f^k(x) - f^{k-1}(x)\big) = b^\delta(x),$$

with some initial guess $f^0(x)$ and regularization parameter α_k, The main ideas of this method can be found in the following references [16], the iteration step is of order $O\left(\delta^{-\frac{\beta}{2p+\beta}}\right)$, which coincides with Landweber iteration [17] and some other iteration methods proposed in [6, 18]. However, with some adaptive choices of regularization parameters $\{\mu_k\}$, the iteration steps can be significantly reduced. For example, the choice of geometric sequence $\{\mu_k\}$ leads to an iteration step of order $O(|\log(\delta)|)$ (see Section 3).

The paper is organized as follows. In Section 2, we introduce paramilitaries and inverse source problems. Section 3 provides the mild solution of problem (1). Section 4, we provide a regularization method and investigate the error between the regularized solution and the exact solution, under a priori and a posteriori parameter choice rule.

2. Preliminaries and inverse source problem

Definition 2.1. The Hilbert scale space $\mathcal{H}^s(\mathbb{X})$, $(s > 0)$ defined by

$$\mathcal{H}^s(\mathbb{X}) := \Big\{f \in L^2(\mathbb{X}) : \sum_{m=1}^{\infty}\xi_m^{2\tau}\big\langle f, \phi_m\big\rangle^2_{L^2(\mathbb{X})} < \infty\Big\},$$

Lemma 2.2. *If $\xi > 0$, then the following equation holds*

$$\int_c^\infty e^{-s\log\frac{t}{b}}\Big(\log\frac{t}{b}\Big)^{\beta k+\gamma-1}E^{(k)}_{\beta,\gamma}\big(\pm\xi\big(\log\frac{t}{c}\big)^\beta\big)\frac{dt}{t} = \frac{k!s^{\beta-\gamma}}{\Big(s^\beta \mp \xi\Big)^{k+1}}, \quad \mathrm{Re}(s) > |\xi|^{\frac{1}{\beta}},$$

where $E^{(m)}_{\beta,\gamma}(y) := \dfrac{d^m}{dy^m}E_{\beta,\gamma}(y)$. *The Lemma 2.2 means that the Laplace transformation of*

$$\big(\log\frac{t}{c}\big)^{\beta k+\gamma-1}E^{(k)}_{\beta,\gamma}\Big(\pm\xi\big(\log\frac{t}{c}\big)^{\beta}\Big) \text{ is } \frac{k!s^{\beta-\gamma}}{(s^{\beta}\mp\xi)^{k+1}}.$$

Lemma 2.3. *For* $0<\gamma<1, z>0$, *we have* $0\le E_{\gamma,1}(-z)<1$. *Moreover,* $E_{\gamma,1}(-z)$ *is completely monotonic, that is,*

$$(-1)^n\frac{d^n}{dz^n}E_{\gamma,1}(-z)\ge 0, z\ge 0,$$

Lemma 2.4. *For a sequence of positive numbers* $\{\alpha_j\}$, *denote* $\sigma_n := \sum_{j=1}^{n}\frac{1}{\alpha_j}$. *Assume that there is a constant* $c>0$ *such that*

$$\frac{1}{\alpha_n}\le c\sigma_{n-1}, \tag{2}$$

for all sufficiently large n. We define a function

$$r_n(\lambda) := \prod_{j=1}^{n}\frac{\alpha_j}{\lambda+\alpha_j},$$

then it holds that

$$\max_{\lambda\in[0,\infty)}\lambda^{\upsilon}r_n(\lambda)\le c_{\upsilon}\sigma_n^{-\upsilon}, \text{where } c_{\upsilon}=\begin{cases}(2\upsilon(c+1))^{\upsilon}, & 0<\upsilon\le 1,\\ \big(2\upsilon(c+1)^{\upsilon}\big)^{\upsilon}, & \upsilon>1.\end{cases}$$

Lemma 2.5. *For any* $\xi_m, s>1$ *that satisfies* $0<\xi_1^s\le\xi_m^s$, *there is a positive number* $\mathcal{B}$ *that depends on* β, c, T, *and* ξ_1 *such that*

$$\frac{\mathcal{Q}_1\mathcal{B}}{\xi_m^{\sigma}}\le\int_b^{\mathcal{T}}\Big(\log\frac{\mathcal{T}}{z}\Big)^{\gamma-1}E_{\gamma,\gamma}\Big(-\xi_m^{\sigma}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma}\Big)\mathcal{Q}(z)\frac{dz}{z}\le\frac{\mathcal{Q}_2}{\xi_m^{\sigma}},$$

where $\mathcal{B}=1-E_{\gamma,1}\big(-\xi_1^{\sigma}\big(\log(\frac{\mathcal{T}}{b})\big)^{\gamma}\big)$.

Proof. See in [13]. □

3. The mild solution

The solution of problem (7) is obtained

$$\begin{aligned}\theta(x,t)=&\sum_{n=1}^{\infty}\Big(E_{\beta,1}\big(-\xi_m^{\sigma}\big(\log\frac{t}{b}\big)^{\gamma}\big)\Big(\int_{\mathbb{X}}a(x)\phi_m(x)dx\Big)\\&+\int_b^t\big(\log\frac{t}{z}\big)^{\gamma-1}E_{\gamma,\gamma}\Big(-\xi_m^{\sigma}\Big(\log\frac{t}{z}\Big)^{\gamma}\Big)\mathcal{Q}(z)\frac{dz}{z}\Big)\Big(\int_{\mathbb{X}}f(x)\phi_m(x)dx\Big)\phi_m(x).\end{aligned}\tag{3}$$

From (3), we know that the Fourier coefficients as follows

$$\begin{aligned}\big\langle\theta(.,t),\phi_m\big\rangle=&\Big(E_{\gamma,1}\big(-\xi_m^{\sigma}\big(\log\frac{t}{b}\big)^{\gamma}\big)\Big(\int_{\mathbb{X}}a(x)\phi_m(x)dx\Big)\\&+\int_b^t\big(\log\frac{t}{z}\big)^{\gamma-1}E_{\gamma,\gamma}\Big(-\xi_m^{\sigma}\big(\log\frac{t}{z}\big)^{\gamma}\Big)\mathcal{Q}(z)\frac{dz}{z}\Big)\Big(\int_{\mathbb{X}}f(x)\phi_m(x)dx\Big).\end{aligned}$$

Using the condition $\int_{\Omega} \theta(x,\mathcal{T})\phi_m(x)dx = b(x)$, we receive

$$\Big(\int_{\Omega} b(x)\phi_m(x)dx\Big) = \Big(E_{\gamma,1}\big(-\xi_m^{\sigma}\big(\log\frac{\mathcal{T}}{b}\big)^{\gamma}\big)\Big)\Big(\int_{\Omega} a(x)\phi_m(x)dx\Big) + \int_{b}^{\mathcal{T}} \big(\log\frac{\mathcal{T}}{z}\big)^{\gamma-1} E_{\gamma,\gamma}\big(-\xi_m^{\sigma}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma}\big)\mathcal{Q}(z)\frac{dz}{z}\ \Big(\int_{\mathbb{X}} f(x)\phi_m(x)dx\Big). \tag{4}$$

Through simple transformations for (4), we get

$$\Big(\int_{\mathbb{X}} f(x)\phi_m(x)dx\Big) = \frac{\int_{\mathbb{X}} b(x)\phi_m(x)dx - \int_{\mathbb{X}} a(x)\phi_m(x)dx\Big(E_{\beta,1}\big(-\xi_m^{s}(\log\frac{\mathcal{T}}{b})^{\gamma}\big)\Big)}{\int_{b}^{\mathcal{T}} \big(\log\frac{\mathcal{T}}{z}\big)^{\gamma-1} E_{\gamma,\gamma}\big(-\xi_m^{\sigma}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma}\big)\mathcal{Q}(z)\frac{dz}{z}}.$$

This leads to

$$f(x) = \sum_{n=1}^{\infty} \frac{\int_{\mathbb{X}} b(x)\phi_m(x)dx - \int_{\mathbb{X}} a(x)\phi_m(x)dx\ E_{\gamma,1}\big(-\xi_m^{\sigma}(\log\frac{\mathcal{T}}{b})^{\gamma}\big)}{\int_{b}^{\mathcal{T}} \big(\log\frac{\mathcal{T}}{z}\big)^{\gamma-1} E_{\gamma,\gamma}\big(-\xi_m^{\sigma}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma}\big)\mathcal{Q}(z)\frac{dz}{z}}\phi_m(x).$$

We denote

$$\int_{\mathbb{X}} \ell(x)\phi_m(x)dx = \int_{\mathbb{X}} b(x)\phi_m(x)dx - \int_{\mathbb{X}} a(x)\phi_m(x)dx\ E_{\gamma,1}\Big(-\xi_m^{\sigma}\big(\log\frac{\mathcal{T}}{b}\big)^{\gamma}\Big).$$

It becomes

$$f(x) = \sum_{n=1}^{\infty} \frac{\int_{\mathbb{X}} \ell(x)\phi_m(x)dx}{\int_{b}^{\mathcal{T}} \big(\log\frac{\mathcal{T}}{z}\big)^{\gamma-1} E_{\gamma,\gamma}\big(-\xi_m^{\sigma}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma}\big)\mathcal{Q}(z)\frac{dz}{z}}\phi_m(x).$$

Defining operator $\mathcal{K} : f \to \ell$, then problem (7) can be converted to the following operator equation: $\mathcal{K}f(x) = \ell(x)$. $\mathcal{J}$ is a bound self-adjoint compact operator, its singular value is

$$\{\sigma_m\}_{m=1}^{\infty}, \Lambda_m = \int_{b}^{\mathcal{T}} \big(\log\frac{\mathcal{T}}{z}\big)^{\gamma-1} E_{\gamma,\gamma}\big(-\xi_m^{\sigma}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma}\big)\mathcal{Q}(z)\frac{dz}{z}. \tag{5}$$

When $m \to \infty, \xi_m \to \infty, \Lambda_m^{-1} \to \infty$, so from formula (5). Hence, the inverse problem is ill-posed, a priori bound condition for the exact solution $f(x)$ as follows:

$$\|f\|_{\mathcal{H}^s(\Omega)} = \Big(\sum_{n=1}^{\infty} \xi_m^{2s}\Big|\langle f, e_m\rangle\Big|^2\Big)^{\frac{1}{2}} \le \mathcal{E},$$

where $\mathcal{C}$ and s are both positive constants.

Theorem 3.1. *Let $\|f\|_{\mathcal{H}^s(\mathbb{X})} \le \mathcal{E}, \mathcal{Q}(t) \in L^{\infty}(b,T)$ and $0 < \mathcal{Q}_1 \le \mathcal{Q}(t) \le \mathcal{Q}_2$. For any $t \in [b,\mathcal{T}]$, we have*

$$\big\|f\big\|_{L^2(\Omega)} \le \Big(\frac{1}{\mathcal{Q}_1\mathcal{B}}\Big)^{\frac{s}{s+1}} \mathcal{E}^{\frac{1}{s+1}} \|\ell\|_{L^2(\mathbb{X})}^{\frac{s}{s+1}}, s > 0. \tag{6}$$

4. A non-stationary iterated quasi-boundary value method

4.1. Well-posedness of the regularization problem

In this work, let $\{\theta^{m,\delta}(x,t), f^{m,\delta}(x)\}$ be the solutions of the following problem:

$$\begin{cases} {}_{CH}\mathbb{D}^{\gamma}_{b,t}\theta^{m,\delta}(x,t) + (-\Delta)^{\sigma}\theta^{m,\delta}(x,t) = \mathcal{F}^{m,\delta}(x)\mathcal{Q}^{\delta}(t), & x \in \mathbb{X}, t \in (b,\mathcal{T}], 0 < \gamma < 1,\ b > 1, \\ \theta^{m,\delta}(x,b) = a^{\delta}(x), & x \in \mathbb{X}, \\ \theta^{m,\delta}(x,t) = 0, & x \in \partial\mathbb{X}, t \in (b,T], \\ \theta^{m,\delta}(x,\mathcal{T}) + \alpha_m\big(f^{m,\delta}(x) - f^{m-1,\delta}(x)\big) = b^{\delta}(x), & x \in \mathbb{X}, \end{cases} \tag{7}$$

for $m = 1, 2, \ldots$, where $f^{0,\delta}(x) = f^0(x)$ is the initial guess of the exact solution $f(x)$ and $\alpha_m > 0$ is the regularization parameter.

Theorem 4.1. *Let $\gamma \in (0,1)$, $\big(a^{\delta}, b^{\delta}\big) \in L^2(\mathbb{X})$, then there exists a unique pair of solutions $f^{\delta}_{\alpha}(x) \in L^2(\mathbb{X}) \times C\big([0,T];\mathbb{H}^s_0(\mathbb{X})\big)$ to problem (7) that can be expressed by*

$$f^{\delta_1}_{\alpha}(x) = \sum_{n=1}^{\infty} \frac{\int\limits_{\mathbb{X}} \ell^{\delta_1}(x)\phi_m(x)dx}{\int\limits_{b}^{\mathcal{T}} \big(\log\frac{\mathcal{T}}{z}\big)^{\gamma-1} E_{\gamma,\gamma}\big(-\xi^{\sigma}_m\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma}\big)\mathcal{Q}(z)\frac{dz}{z} + \alpha}\phi_m(x), \tag{8}$$

and

$$f^{\delta_2}_{\alpha}(x) = \sum_{n=1}^{\infty} \frac{\int\limits_{\mathbb{X}} \ell^{\delta_2}(x)\phi_m(x)dx}{\int\limits_{b}^{\mathcal{T}} \big(\log\frac{\mathcal{T}}{z}\big)^{\gamma-1} E_{\gamma,\gamma}\big(-\xi^{\sigma}_m\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma}\big)\mathcal{Q}(z)\frac{dz}{z} + \alpha}\phi_m(x). \tag{9}$$

Theorem 4.2. *Let $\ell^{\delta_1}, \ell^{\delta_2} \in L^2(\mathbb{X})$, we denote $(f^{\delta_1}_{\alpha}(x))$ and $(f^{\delta_2}_{\alpha}(x))$ as the solutions of (8)-(9) with respect to ℓ^{δ_1} and ℓ^{δ_2}, then we have*

$$\big\|f^{\delta_1}_{\alpha} - f^{\delta_2}_{\alpha}\big\|_{L^2(\mathbb{X})} \le \alpha^{-1}\big\|\ell^{\delta_1} - \ell^{\delta_2}\big\|_{L^2(\mathbb{X})}.$$

Proof. It follows from (8) and (9) that

$$f^{\delta_1}_{\alpha}(x) - f^{\delta_2}_{\alpha}(x) = \sum_{n=1}^{\infty} \frac{\int\limits_{\mathbb{X}} \big(\ell^{\delta_1}(x) - \ell^{\delta_2}(x)\big)\phi_m(x)dx}{\int\limits_{b}^{\mathcal{T}} \big(\log\frac{\mathcal{T}}{z}\big)^{\gamma-1} E_{\gamma,\gamma}\big(-\xi^{\sigma}_m\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma}\big)\mathcal{Q}(z)\frac{dz}{z} + \alpha}\phi_m(x).$$

Thus it can be deduced that

$$\begin{aligned} \big\|f^{\delta_1}_{\alpha} - f^{\delta_2}_{\alpha}\big\|^2_{L^2(\mathbb{X})} &= \sum_{n=1}^{\infty} \Big(\int_{b}^{\mathcal{T}} \big(\log\frac{\mathcal{T}}{z}\big)^{\gamma-1} E_{\gamma,\gamma}\big(-\xi^{\sigma}_m\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma}\big)\mathcal{Q}(z)\frac{dz}{z} + \alpha\Big)^{-2} \\ &\qquad \times \Big(\int_{\mathbb{X}} \big(\ell^{\delta_1}(x) - \ell^{\delta_2}(x)\big)\phi_m(x)dx\Big)^2 \\ &\le \frac{1}{\alpha^2}\big\|\ell^{\delta_1} - \ell^{\delta_2}\big\|^2_{L^2(\mathbb{X})}. \end{aligned}$$

□

5. Convergence Analysis

Theorem 5.1. *Assume that $\mathcal{Q} \in L^\infty(0,\mathcal{T})$ satisfies $\mathcal{Q}_1 \le \mathcal{Q}(t) \le \mathcal{Q}_2$. Let $f - f^0 \in \mathbb{H}_0^s(\mathbb{X})$ satisfy an a priori bound condition*

$$\big\|f - f^0\big\|_{\mathbb{H}_0^s(\mathbb{X})} \le \mathcal{D}, \quad s \ge 0$$

then we have

$$\big\|f - f^0\big\|_{L^2(\mathbb{X})} \le \big(\mathcal{Q}_1\mathcal{B}\big)^{-\frac{4s}{2s+\sigma}} \mathcal{D}^{\frac{\sigma}{2s+\sigma}} \big\|\mathcal{K}\big(f - f^0\big)\big\|^{\frac{2s}{2s+\sigma}}, \quad s \ge 0,$$

where $\mathcal{D} = \mathcal{D}\big(\alpha, \gamma, \mathcal{Q}_1, s, \mathcal{T}, \xi_1\big)$.

Proof. It follows that

$$\mathcal{K}\big(f - f^0\big) = \sum_{n=1}^{\infty} \int_b^{\mathcal{T}} \big(\log \frac{\mathcal{T}}{z}\big)^{\gamma-1} E_{\gamma,\gamma}\big(-\xi_m^\sigma \big(\log \frac{\mathcal{T}}{z}\big)^\gamma\big) \mathcal{Q}(z) \frac{dz}{z} \times \Big(\int_{\mathbb{X}} f(x)\phi_m(x)dx - \int_{\mathbb{X}} f^0(x)\phi_m(x)dx\Big)\phi_m(x).$$

We have

$$\begin{aligned}
&\big\|f - f^0\big\|_{L^2(\Omega)}^2 \\
&= \sum_{n=1}^{\infty} \Big(\int_{\mathbb{X}} f(x)\phi_m(x)dx - \int_{\mathbb{X}} f^0(x)\phi_m(x)dx\Big)^2 \\
&= \sum_{n=1}^{\infty} \frac{\Big[\int_b^{\mathcal{T}} \big(\log \frac{\mathcal{T}}{z}\big)^{\gamma-1} E_{\gamma,\gamma}\big(-\xi_m^\sigma \big(\log \frac{\mathcal{T}}{z}\big)^\gamma\big) \mathcal{Q}(z) \frac{dz}{z} \Big(\int_{\mathbb{X}} f(x)\phi_m(x)dx - \int_{\mathbb{X}} f^0(x)\phi_m(x)dx\Big)\Big]^2}{\Big|\int_b^{\mathcal{T}} \big(\log \frac{\mathcal{T}}{z}\big)^{\gamma-1} E_{\gamma,\gamma}\big(-\xi_m^\sigma \big(\log \frac{\mathcal{T}}{z}\big)^\gamma\big) \mathcal{Q}(z) \frac{dz}{z}\Big|^2} \\
&= \sum_{n=1}^{\infty} \frac{\Big[\int_b^{\mathcal{T}} \big(\log \frac{\mathcal{T}}{z}\big)^{\gamma-1} E_{\gamma,\gamma}\big(-\xi_m^\sigma \big(\log \frac{\mathcal{T}}{z}\big)^\gamma\big) \mathcal{Q}(z) \frac{dz}{z} \Big(\int_{\mathbb{X}} f(x)\phi_m(x)dx - \int_{\mathbb{X}} f^0(x)\phi_m(x)dx\Big)\Big]^{\frac{2\sigma}{2s+\sigma}}}{\Big[\int_b^{\mathcal{T}} \big(\log \frac{\mathcal{T}}{z}\big)^{\gamma-1} E_{\gamma,\gamma}\big(-\xi_m^\sigma \big(\log \frac{\mathcal{T}}{z}\big)^\gamma\big) \mathcal{Q}(z) \frac{dz}{z}\Big]^2} \\
&\quad \times \Big[\int_b^{\mathcal{T}} \big(\log \frac{\mathcal{T}}{z}\big)^{\gamma-1} E_{\gamma,\gamma}\big(-\xi_m^\sigma \big(\log \frac{\mathcal{T}}{z}\big)^\gamma\big) \mathcal{Q}(z) \frac{dz}{z} \Big(\int_{\mathbb{X}} f(x)\phi_m(x)dx - \int_{\mathbb{X}} f^0(x)\phi_m(x)dx\Big)\Big]^{\frac{4s}{2s+\sigma}}.
\end{aligned}$$

Thanks to Hölder's inequality, we obtain

$$\begin{aligned}
&\big\|f - f^0\big\|_{L^2(\Omega)}^2 \\
&\le \Big(\sum_{n=1}^{\infty} \Big[\Big|\int_b^{\mathcal{T}} \big(\log \frac{\mathcal{T}}{z}\big)^{\gamma-1} E_{\gamma,\gamma}\big(-\xi_m^\sigma \big(\log \frac{\mathcal{T}}{z}\big)^\gamma\big) \mathcal{Q}(z) \frac{dz}{z}\Big|^{-\frac{2s}{\sigma}} \Big(\int_{\mathbb{X}} \big(f(x) - f^0(x)\big)\phi_m(x)dx\Big)\Big]^2\Big)^{\frac{\sigma}{2s+\sigma}} \\
&\quad \times \Big(\sum_{n=1}^{\infty} \Big[\int_b^{\mathcal{T}} \big(\log \frac{\mathcal{T}}{z}\big)^{\gamma-1} E_{\gamma,\gamma}\big(-\xi_m^\sigma \big(\log \frac{\mathcal{T}}{z}\big)^\gamma\big) \mathcal{Q}(z) \frac{dz}{z} \Big(\int_{\mathbb{X}} \Big(f(x) - f^0(x)\Big)\phi_m(x)dx\Big)\Big]^2\Big)^{\frac{2s}{2s+\sigma}}.
\end{aligned}$$

This deduce

$$\begin{aligned}
&\big\|f-f^0\big\|^2_{L^2(\Omega)}\\
&\leq\big(\mathcal{Q}_1\mathcal{B}\big)^{-\frac{4s}{2s+\sigma}}\Big[\sum_{n=1}^{\infty}\xi_m^{2s}\Big(\int_{\mathbb{X}}\big(f(x)-f^0(x)\big)\phi_m(x)dx\Big)^2\Big]^{\frac{\sigma}{2s+\sigma}}\\
&\quad\times\Big(\sum_{n=1}^{\infty}\Big[\int_b^{\mathcal{T}}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma-1}E_{\gamma,\gamma}\big(-\xi_m^{\sigma}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma}\big)\mathcal{Q}(z)\frac{dz}{z}\\
&\quad\times\Big(\int_{\mathbb{X}}\big(f(x)-f^0(x)\big)\phi_m(x)dx\Big)\Big]^2\Big)^{\frac{2s}{2s+\sigma}}\\
&=\big(\mathcal{Q}_1\mathcal{B}\big)^{-\frac{4s}{2s+\sigma}}\big\|f-f^0\big\|^{\frac{2\sigma}{2s+\sigma}}_{\mathbb{H}_0^{2s}(\mathbb{X})}\big\|\mathcal{K}(f-f^0)\big\|^{\frac{4s}{2s+\sigma}}\\
&\leq\big(\mathcal{Q}_1\mathcal{B}\big)^{-\frac{4s}{2s+\sigma}}\mathcal{D}^{\frac{2\sigma}{2s+\sigma}}\big\|\mathcal{K}\big(f-f^0\big)\big\|^{\frac{4s}{2s+\sigma}}.
\end{aligned}$$

Then the proof is completed. □

5.1. An a priori parameter choice rule.

By Theorem 4.1, the reconstructed source $f^{k,\delta}(x)$ can be expressed by

$$\begin{aligned}
f^{k,\delta}(x)=&\sum_{n=1}^{\infty}\frac{\alpha_m}{\int_b^{\mathcal{T}}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma-1}E_{\gamma,\gamma}\big(-\xi_m^{\sigma}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma}\big)\mathcal{Q}(z)\frac{dz}{z}+\alpha_m}\Big(\int_{\mathbb{X}}f^{k-1,\delta}(x)\phi_m(x)dx\Big)\phi_m(x)\\
&+\sum_{n=1}^{\infty}\frac{1}{\int_b^{\mathcal{T}}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma-1}E_{\gamma,\gamma}\big(-\xi_m^{\sigma}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma}\big)\mathcal{Q}(z)\frac{dz}{z}+\alpha_m}\Big(\int_{\mathbb{X}}\ell^{\delta}(x)\phi_m(x)dx\Big)\phi_m(x).
\end{aligned}$$

By a recursive approach, the above expression can be rewritten as

$$\begin{aligned}
f^{k,\delta}(x)=&\sum_{n=1}^{\infty}r_k\Big(\int_b^{\mathcal{T}}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma-1}E_{\gamma,\gamma}\big(-\xi_m^{\sigma}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma}\big)\mathcal{Q}(z)\frac{dz}{z}\Big)\Big(\int_{\mathbb{X}}f^0(x)\phi_m(x)dx\Big)\phi_m(x)\\
&+\sum_{n=1}^{\infty}s_k\Big(\int_b^{\mathcal{T}}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma-1}E_{\gamma,\gamma}\big(-\xi_m^{\sigma}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma}\big)\mathcal{Q}(z)\frac{dz}{z}\Big)\Big(\int_{\mathbb{X}}\ell^{\delta}(x)\phi_m(x)dx\Big)\phi_m(x),
\end{aligned}$$

for $f^0(x)=f^{0,\delta}(x)$, where $r_k(\xi)$ is defined in Lemma 2.4 and $s_k(\xi)$ is generated by

$$s_k(\xi)=\frac{1+\alpha_k s_{k-1}(\xi)}{\xi+\alpha_k},\quad s_0(\xi)=0.$$

For the function $s_k(\xi)$, we have the following result.

Lemma 5.2. *For all $\xi\geq 0$, we have $s_m(\xi)\leq\varsigma_m$.*

Theorem 5.3. *Assume that $\mathcal{Q}\in L^{\infty}(0,\mathcal{T})$. Let $\{\alpha_j\}\subset\mathbb{R}^+$ be a sequence satisfying condition (2), and f^0 be an initial guess of f satisfying the a priori condition (6). We choose the integer $k(\mathcal{E},\delta)$ satisfying $\varsigma^{-1}_{k(E,\delta)}=O\big(\big(\frac{\delta}{E}\big)^{\frac{\sigma}{2s+\sigma}}\big)$, then there exists a positive constant $\mathcal{C}$ depends on $\alpha,\sigma,s,\mathcal{Q}_1,\mathcal{T},\xi_1$ such that*

$$\big\|f^{k(\delta),\delta}-f\big\|_{L^2(\mathbb{X})}\leq C\mathcal{D}^{\frac{\sigma}{2s+\sigma}}\delta^{\frac{2s}{2s+\sigma}}. \tag{10}$$

Proof. We have

$$\left\|f^{k,\delta}-f\right\|_{L^2(\mathbb{X})}\leq\left\|f^{k,\delta}-f^k\right\|_{L^2(\mathbb{X})}+\left\|f^k-f\right\|_{L^2(\mathbb{X})}. \tag{11}$$

Next, we will divide it into two steps to accomplish the proof.

Claim 1: We consider $\left\|f^k-f\right\|_{L^2(\mathbb{X})}$. Firstly, for exact $\ell(x), f^k(x)$ can be expressed by

$$\begin{aligned}f^k(x)=&\sum_{n=1}^{\infty}r_k\Big(\int_b^{\mathcal{T}}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma-1}E_{\gamma,\gamma}\big(-\xi_m^{\sigma}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma}\big)\mathcal{Q}(z)\frac{dz}{z}\Big)\Big(\int_{\mathbb{X}}f^0(x)\phi_m(x)dx\Big)\phi_m(x)\\&+\sum_{n=1}^{\infty}s_k\big(\int_b^{\mathcal{T}}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma-1}E_{\gamma,\gamma}\big(-\xi_m^{\sigma}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma}\big)\mathcal{Q}(z)\frac{dz}{z}\big)\Big(\int_{\mathbb{X}}\ell(x)\phi_m(x)dx\Big)\phi_m(x),\end{aligned}$$

then we have

$$\begin{aligned}&f^k(x)-f(x)\\&=\sum_{n=1}^{\infty}r_k\big(\int_b^{\mathcal{T}}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma-1}E_{\gamma,\gamma}\big(-\xi_m^{\sigma}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma}\big)\mathcal{Q}(z)\frac{dz}{z}\big)\Big(\int_{\mathbb{X}}f^0(x)\phi_m(x)dx\Big)\phi_m(x)\\&\quad+\sum_{n=1}^{\infty}\Big(s_k\Big(\int_b^{\mathcal{T}}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma-1}E_{\gamma,\gamma}\big(-\xi_m^{\sigma}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma}\big)\mathcal{Q}(z)\frac{dz}{z}\Big)\\&\quad-\frac{\int_{\mathbb{X}}\ell(x)\phi_m(x)dx}{\int_b^{\mathcal{T}}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma-1}E_{\gamma,\gamma}\big(-\xi_m^{\sigma}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma}\big)\mathcal{Q}(z)\frac{dz}{z}}\Big)\phi_m(x)\\&=\sum_{n=1}^{\infty}r_k\big(\int_b^{\mathcal{T}}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma-1}E_{\gamma,\gamma}\big(-\xi_m^{\sigma}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma}\big)\mathcal{Q}(z)\frac{dz}{z}\big)\Big(\int_{\mathbb{X}}\big(f^0(x)-f(x)\big)\phi_m(x)dx\Big)\phi_m(x).\end{aligned} \tag{12}$$

From (12), it yields

$$\begin{aligned}&\left\|f^k-f\right\|_{L^2(\mathbb{X})}^2\\&=\sum_{n=1}^{\infty}r_k^2\big(\int_b^{\mathcal{T}}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma-1}E_{\gamma,\gamma}\big(-\xi_m^{\sigma}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma}\big)\mathcal{Q}(z)\frac{dz}{z}\big)\Big(\int_{\mathbb{X}}\big(f^0(x)-f(x)\big)\phi_m(x)dx\Big)^2\\&=\sum_{n=1}^{\infty}\Big|\int_b^{\mathcal{T}}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma-1}E_{\gamma,\gamma}\big(-\xi_m^{\sigma}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma}\big)\mathcal{Q}(z)\frac{dz}{z}\Big|^{\frac{4s}{\sigma}}r_k^2\\&\quad\times\Big|\int_b^{\mathcal{T}}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma-1}E_{\gamma,\gamma}\big(-\xi_m^{\sigma}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma}\big)\mathcal{Q}(z)\frac{dz}{z}\Big|\\&\quad\times\Big|\int_b^{\mathcal{T}}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma-1}E_{\gamma,\gamma}\big(-\xi_m^{\sigma}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma}\big)\mathcal{Q}(z)\frac{dz}{z}\Big|^{-\frac{4s}{\sigma}}\Big(\int_{\mathbb{X}}\big(f^0(x)-f(x)\big)\phi_m(x)dx\Big)^2.\end{aligned} \tag{13}$$

From (13), we know that

$$\Big|\int_b^{\mathcal{T}}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma-1}E_{\gamma,\gamma}\big(-\xi_m^{\sigma}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma}\big)\mathcal{Q}(z)\frac{dz}{z}\Big|^{\frac{2s}{\sigma}}r_k$$
$$\times\Big|\int_b^{\mathcal{T}}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma-1}E_{\gamma,\gamma}\big(-\xi_m^{\sigma}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma}\big)\mathcal{Q}(z)\frac{dz}{z}\Big|\le\mathcal{D}(s,\sigma)\varsigma_k^{-\frac{2s}{\sigma}}.$$

This leads to

$$\begin{aligned}\big\|f^k-f\big\|^2_{L^2(\mathbb{X})}&\le\mathcal{D}^2(s,\sigma)(\mathcal{Q}_1\mathcal{B})^{-\frac{4s}{\sigma}}\varsigma_k^{-\frac{4s}{\sigma}}\sum_{n=1}^{\infty}\xi_m^{2s}\Big(\int_{\mathbb{X}}\big(f^0(x)-f(x)\big)\phi_m(x)dx\Big)^2\\&=\mathcal{C}\big\|f^0-f\big\|^2_{\mathbb{H}_0^{2s}(\mathbb{X})}\varsigma_k^{-\frac{4s}{\sigma}}\le C\mathcal{E}^2\varsigma_k^{-\frac{4s}{\sigma}}.\end{aligned}\tag{14}$$

Claim 2 Next, we consider $\big\|f^{k,\delta}-f^k\big\|_{L^2(\mathbb{X})}$. It follows from Lemma 3.5 that

$$\begin{aligned}\big\|f^{k,\delta}-f^k\big\|^2_{L^2(\mathbb{X})}=&\sum_{n=1}^{\infty}s_k^2\Big(\int_b^{\mathcal{T}}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma-1}E_{\gamma,\gamma}\big(-\xi_m^{\sigma}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma}\big)\mathcal{Q}(z)\frac{dz}{z}\Big)\\&\times\Big(\int_{\mathbb{X}}\big(\ell^{\delta}(x)-\ell(x)\big)\phi_m(x)dx\Big)^2\le\sigma_k^2\delta^2(1+\overline{\mathcal{C}})^2.\end{aligned}\tag{15}$$

If the integer $k(\mathcal{E},\delta)$ satisfies $\sigma^{-1}_{k(\mathcal{E},\delta)}=O\Big(\Big(\frac{\delta}{E}\Big)^{\frac{\sigma}{2s+\sigma}}\Big)$, combining (14) and (15) directly gives the estimate (10). Then the proof is completed. □

5.2. An a-posteriori parameter choice rule

In this case, we intend to adopt the discrepancy principle to terminate the iteration, i.e. we choose a value $k=k(\boldsymbol{\delta})$ as long as it satisfies:

$$\big\|\mathcal{K}f^{k(\delta),\delta}-\ell^{\delta}\big\|_{L^2(\mathbb{X})}\le\tau\delta,\tag{16}$$

for some constant $\tau>1$. The following lemma ensures that we can obtain a finite number $k(\delta)$ from (16).

Lemma 5.4. *Assume that $f^0,f,g^{\delta}\in L^2(\Omega)$ and the following conditions hold:*

- $\big\|\mathcal{K}f^0-\ell^{\delta}\big\|_{L^2(\Omega)}\ge\tau\delta,\quad$ *for some $\tau>1$.*
- $\lim_{k\to\infty}\varsigma_k=\infty$, *then for sufficiently large $k\in\mathbb{N}^+$, it holds that*

$$\big\|\mathcal{K}f^{k,\delta}-\ell^{\delta}\big\|_{L^2(\mathbb{X})}\le\tau\delta.$$

Proof. We can deduce that

$$\begin{aligned}&\mathcal{K}f^{k,\delta}-\ell^{\delta}\\&=\sum_{n=1}^{\infty}\int_b^{\mathcal{T}}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma-1}E_{\gamma,\gamma}\big(-\xi_m^{\sigma}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma}\big)\mathcal{Q}(z)\frac{dz}{z}\\&\quad\times r_k\Big(\int_b^{\mathcal{T}}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma-1}E_{\gamma,\gamma}\big(-\xi_m^{\sigma}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma}\big)\mathcal{Q}(z)\frac{dz}{z}\Big)\Big(\int_{\mathbb{X}}f^0(x)\phi_m(x)dx\Big)\phi_m(x)\end{aligned}$$

and then

$$\begin{aligned}
&\mathcal{K}f^{k,\delta}-\ell^{\delta}\\
&\quad=\sum_{n=1}^{\infty}\Big[\int_{b}^{\mathcal{T}}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma-1}E_{\gamma,\gamma}\big(-\xi_m^{\sigma}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma}\big)\mathcal{Q}(z)\frac{dz}{z}\\
&\qquad\times r_k\Big(\int_{b}^{\mathcal{T}}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma-1}E_{\gamma,\gamma}\big(-\xi_m^{\sigma}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma}\big)\mathcal{Q}(z)\frac{dz}{z}\Big)-1\Big]\\
&\qquad\times\Big(\int_{\mathbb{X}}\ell^{\delta}(x)\phi_m(x)dx\Big)\phi_m(x)
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{K}f^{k,\delta}-\ell^{\delta}\\
&\quad=\sum_{n=1}^{\infty}\Big|\int_{b}^{\mathcal{T}}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma-1}E_{\gamma,\gamma}\big(-\xi_m^{\sigma}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma}\big)\mathcal{Q}(z)\frac{dz}{z}\Big|\\
&\qquad\times r_k\Big(\int_{b}^{\mathcal{T}}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma-1}E_{\gamma,\gamma}\big(-\xi_m^{\sigma}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma}\big)\mathcal{Q}(z)\frac{dz}{z}\Big)\\
&\qquad\times\Big(\int_{\mathbb{X}}f^{0}(x)\phi_m(x)dx\Big)\phi_m(x)\\
&\qquad-\sum_{n=1}^{\infty}r_k\Big(\int_{b}^{\mathcal{T}}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma-1}E_{\gamma,\gamma}\big(-\xi_m^{\sigma}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma}\big)\mathcal{Q}(z)\frac{dz}{z}\Big)\Big(\int_{\mathbb{X}}\ell^{\delta}(x)\phi_m(x)dx\Big)\phi_m(x).
\end{aligned}$$

So

$$\begin{aligned}
&\mathcal{K}f^{k,\delta}-\ell^{\delta}\\
&\quad=\sum_{n=1}^{\infty}r_k\Big|\int_{b}^{\mathcal{T}}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma-1}E_{\gamma,\gamma}\big(-\xi_m^{\sigma}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma}\big)\mathcal{Q}(z)\frac{dz}{z}\Big|\\
&\qquad\times\Big(\int_{b}^{\mathcal{T}}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma-1}E_{\gamma,\gamma}\big(-\xi_m^{\sigma}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma}\big)\mathcal{Q}(z)\frac{dz}{z}\int_{\mathbb{X}}f^{0}(x)\phi_m(x)dx-\int_{\mathbb{X}}\ell^{\delta}(x)\phi_m(x)dx\Big)\phi_m(x)\\
&\qquad=\sum_{n=1}^{\infty}r_k\Big(\int_{b}^{\mathcal{T}}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma-1}E_{\gamma,\gamma}\big(-\xi_m^{\sigma}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma}\big)\mathcal{Q}(z)\frac{dz}{z}\Big)\\
&\qquad\qquad\times\int_{b}^{\mathcal{T}}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma-1}E_{\gamma,\gamma}\big(-\xi_m^{\sigma}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma}\big)\mathcal{Q}(z)\frac{dz}{z}\Big(\int_{\mathbb{X}}(f^{0}(x)-f(x))\phi_m(x)dx\Big)\phi_m(x)\\
&\qquad+\sum_{n=1}^{\infty}r_k\Big(\int_{b}^{\mathcal{T}}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma-1}E_{\gamma,\gamma}\big(-\xi_m^{\sigma}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma}\big)\mathcal{Q}(z)\frac{dz}{z}\Big)\Big(\int_{\mathbb{X}}\big(\ell(x)-\ell^{\delta}(x)\big)\phi_m(x)dx\Big)\phi_m(x)\\
&\qquad=:\mathcal{I}_1+\mathcal{I}_2.
\end{aligned}$$

Since $\sigma_k \to \infty$ as $k \to \infty$, for $\forall \mu > 0$, there exists a large enough positive number $N \in \mathbb{N}^+$ such that for $k > N$, it yields $\sigma_k^{-1} \leq \mu$. Then

$$\begin{aligned}
\|\mathcal{I}_1\|^2_{L^2(\mathbb{X})} &= \sum_{n=1}^{\infty} \Big| \int_b^{\mathcal{T}} \big(\log \frac{\mathcal{T}}{z}\big)^{\gamma-1} E_{\gamma,\gamma}\big(-\xi_m^{\sigma}\big(\log \frac{\mathcal{T}}{z}\big)^{\gamma}\big)\mathcal{Q}(z)\frac{dz}{z}\Big|^2 \\
&\qquad \times r_k^2 \Big| \int_b^{\mathcal{T}} \big(\log \frac{\mathcal{T}}{z}\big)^{\gamma-1} E_{\gamma,\gamma}\big(-\xi_m^{\sigma}\big(\log \frac{\mathcal{T}}{z}\big)^{\gamma}\big)\mathcal{Q}(z)\frac{dz}{z}\Big| \\
&\qquad \times \Big(\int_{\mathbb{X}} \big(f^0(x) - f(x)\big)\phi_m(x)dx\Big)^2 \\
&\leq \mathcal{C}^2 \varsigma_k^{-2} \sum_{n=1}^{\infty} \Big(\int_{\mathbb{X}} \big(f^0(x) - f(x)\big)\phi_m(x)dx\Big)^2 \\
&\leq C\big\|f^0 - f\big\|^2_{L^2(\mathbb{X})}\mu^2.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\big\|\mathcal{K}f^{k,\delta} - \ell^{\delta}\big\|_{L^2(\mathbb{X})} &\leq \big\|\mathcal{I}_1\big\|_{L^2(\mathbb{X})} + \big\|\mathcal{I}_2\big\|_{L^2(\mathbb{X})} \\
&\leq C\big\|f^0 - f\big\|_{L^2(\mathbb{X})}\mu + \delta(1 + \overline{C}).
\end{aligned}$$

Since μ is arbitrarily chosen, we can finally obtain (16). □

Lemma 5.5. *If $k(\delta)$ is chosen by (16), then*

$$\big\|\mathcal{K}f^{k(\delta)} - \ell\big\|_{L^2(\mathbb{X})}(\tau+1)^{-1} \leq \delta < \big\|\mathcal{K}f^{k(\delta)-1} - g\big\|_{L^2(\mathbb{X})}\big(\tau - 1\big)^{-1}.$$

Theorem 5.6. *We assume that $\mathcal{Q}(t) \in L^{\infty}[b,T]$. Let $\{\alpha_j\} \subset \mathbb{R}^+$ be a sequence satisfying condition (2), and f^0 be an initial guess of f satisfying the a priori condition (6). Let $k(\delta)$ be chosen by (16), then there exists a positive constant $C = C\big(\alpha, \beta, q_0, p, T, \xi_1, \tau\big)$ such that*

$$\big\|f^{k(\delta),\delta} - f\big\| \leq CE^{\frac{\beta}{2p+\beta}}\delta^{\frac{2p}{2p+\beta}}. \tag{17}$$

Proof. We have

$$\big\|f^{k,\delta} - f\big\|_{L^2(\mathbb{X})} \leq \big\|f^{k,\delta} - f^k\big\|_{L^2(\mathbb{X})} + \big\|f^k - f\big\|_{L^2(\mathbb{X})}.$$

Step 1: Estimate of $\big\|f^k - f\big\|_{L^2(\mathbb{X})}$,

$$\begin{aligned}
f^k(x) &= \sum_{n=1}^{\infty} r_k\Big(\int_b^{\mathcal{T}} \big(\log \frac{\mathcal{T}}{z}\big)^{\gamma-1} E_{\gamma,\gamma}\big(-\xi_m^{\sigma}\big(\log \frac{\mathcal{T}}{z}\big)^{\gamma}\big)\mathcal{Q}(z)\frac{dz}{z}\Big) \int_{\mathbb{X}} f^0(x)\phi_m(x)dx\ \phi_m(x) \\
&\quad + \sum_{n=1}^{\infty} s_k\Big(\int_b^{\mathcal{T}} \big(\log \frac{\mathcal{T}}{z}\big)^{\gamma-1} E_{\gamma,\gamma}\big(-\xi_m^{\sigma}\big(\log \frac{\mathcal{T}}{z}\big)^{\gamma}\big)\mathcal{Q}(z)\frac{dz}{z}\Big) \int_{\mathbb{X}} \ell(x)\phi_m(x)dx\ \phi_m(x),
\end{aligned}$$

then

$$
\begin{aligned}
f^k(x)-f(x)=&\sum_{n=1}^{\infty} r_k\Big|\int_b^{\mathcal{T}}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma-1}E_{\gamma,\gamma}\big(-\xi_m^{\sigma}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma}\big)\mathcal{Q}(z)\frac{dz}{z}\Big|\Big(\int_{\mathbb{X}} f^0(x)\phi_m(x)dx\Big)\phi_m(x)\\
&+\sum_{n=1}^{\infty}\Big(s_k\Big(\int_b^{\mathcal{T}}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma-1}E_{\gamma,\gamma}\big(-\xi_m^{\sigma}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma}\big)\mathcal{Q}(z)\frac{dz}{z}\Big)\\
&-\frac{1}{\Big|\int_b^{\mathcal{T}}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma-1}E_{\gamma,\gamma}\big(-\xi_m^{\sigma}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma}\big)\mathcal{Q}(z)\frac{dz}{z}\Big|}\Big)\\
&\times\Big|\int_b^{\mathcal{T}}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma-1}E_{\gamma,\gamma}\big(-\xi_m^{\sigma}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma}\big)\mathcal{Q}(z)\frac{dz}{z}\Big|\phi_m(x)\\
=&\sum_{n=1}^{\infty} r_k\Big|\int_b^{\mathcal{T}}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma-1}E_{\gamma,\gamma}\big(-\xi_m^{\sigma}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma}\big)\mathcal{Q}(z)\frac{dz}{z}\Big|\times\Big(\int_{\mathbb{X}}\big(f^0(x)-f(x)\big)\phi_m(x)dx\Big)\phi_m(x).
\end{aligned}
$$

Next, we know that

$$
\begin{aligned}
\mathcal{K}\big(f^k(x)-f(x)\big)=&\sum_{n=1}^{\infty}\Big|\int_b^{\mathcal{T}}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma-1}E_{\gamma,\gamma}\big(-\xi_m^{\sigma}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma}\big)\mathcal{Q}(z)\frac{dz}{z}\Big|\\
&\times r_k\Big|\int_b^{\mathcal{T}}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma-1}E_{\gamma,\gamma}\big(-\xi_m^{\sigma}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma}\big)\mathcal{Q}(z)\frac{dz}{z}\Big|\\
&\times\Big(\int_{\mathbb{X}} f^0(x)\phi_m(x)dx\Big)\phi_m(x)\\
&+\sum_{n=1}^{\infty}\Big(\int_b^{\mathcal{T}}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma-1}E_{\gamma,\gamma}\big(-\xi_m^{\sigma}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma}\big)\mathcal{Q}(z)\frac{dz}{z}\\
&\times s_k\Big(\int_b^{\mathcal{T}}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma-1}E_{\gamma,\gamma}\big(-\xi_m^{\sigma}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma}\big)\mathcal{Q}(z)\frac{dz}{z}\Big)-1\Big)\\
&\times\Big(\int_{\mathbb{X}}\ell(x)\phi_m(x)dx\Big)\phi_m(x)\\
&+\sum_{n=1}^{\infty}\Big(\int_b^{\mathcal{T}}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma-1}E_{\gamma,\gamma}\big(-\xi_m^{\sigma}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma}\big)\mathcal{Q}(z)\frac{dz}{z}\\
&\times s_k\Big(\int_b^{\mathcal{T}}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma-1}E_{\gamma,\gamma}\big(-\xi_m^{\sigma}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma}\big)\mathcal{Q}(z)\frac{dz}{z}\Big)-1\Big)\\
&\times\Big(\int_{\mathbb{X}}\ell^{\delta}(x)\phi_m(x)dx\Big)\phi_m(x).
\end{aligned}
\tag{18}
$$

After that, from (18), we get

$$\begin{aligned}
&\sum_{n=1}^{\infty}\Big(\int_b^{\mathcal{T}}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma-1}E_{\gamma,\gamma}\big(-\xi_m^{\sigma}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma}\big)\mathcal{Q}(z)\frac{dz}{z}\\
&\qquad\times s_k\Big|\int_b^{\mathcal{T}}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma-1}E_{\gamma,\gamma}\big(-\xi_m^{\sigma}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma}\big)\mathcal{Q}(z)\frac{dz}{z}\Big|-1\Big)\\
&\qquad\times\Big(\int_{\mathbb{X}}\big(\ell^{\delta}(x)-\ell(x)\big)\phi_m(x)dx\Big)\phi_m(x)\\
=&\mathcal{K}f^{k,\delta}(x)-\ell^{\delta}(x)\\
&+\sum_{n=1}^{\infty}\Big(\int_b^{\mathcal{T}}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma-1}E_{\gamma,\gamma}\big(-\xi_m^{\sigma}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma}\big)\mathcal{Q}(z)\frac{dz}{z}\\
&\qquad\times s_k\Big(\int_b^{\mathcal{T}}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma-1}E_{\gamma,\gamma}\big(-\xi_m^{\sigma}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma}\big)\mathcal{Q}(z)\frac{dz}{z}\Big)-1\Big)\\
&\qquad\times\Big(\int_{\mathbb{X}}\big(\ell^{\delta}(x)-\ell(x)\big)\phi_m(x)dx\Big)\phi_m(x).
\end{aligned}$$

It can be learned that

$$\begin{aligned}
&\int_b^{\mathcal{T}}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma-1}E_{\gamma,\gamma}\big(-\xi_m^{\sigma}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma}\big)\mathcal{Q}(z)\frac{dz}{z}\\
&\times s_k\Big(\int_b^{\mathcal{T}}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma-1}E_{\gamma,\gamma}\big(-\xi_m^{\sigma}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma}\big)\mathcal{Q}(z)\frac{dz}{z}\Big)-1\\
&=-r_k\Big(\int_b^{\mathcal{T}}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma-1}E_{\gamma,\gamma}\big(-\xi_m^{\sigma}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma}\big)\mathcal{Q}(z)\frac{dz}{z}\Big).
\end{aligned}$$

Thus we have

$$\begin{aligned}
&\Big\|\sum_{n=1}^{\infty}\Big(\int_b^{\mathcal{T}}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma-1}E_{\gamma,\gamma}\big(-\xi_m^{\sigma}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma}\big)\mathcal{Q}(z)\frac{dz}{z}\\
&\qquad\times s_k\Big(\int_b^{\mathcal{T}}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma-1}E_{\gamma,\gamma}\big(-\xi_m^{\sigma}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma}\big)\mathcal{Q}(z)\frac{dz}{z}\Big)-1\Big)\\
&\qquad\times\Big(\int_{\mathbb{X}}\big(\ell^{\delta}(x)-\ell(x)\big)\phi_m(x)dx\Big)\phi_m\Big\|^2_{L^2(\mathbb{X})}\\
&\le\sum_{n=1}^{\infty}r_k^2\Big(\int_b^{\mathcal{T}}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma-1}E_{\gamma,\gamma}\big(-\xi_m^{\sigma}\big(\log\frac{\mathcal{T}}{z}\big)^{\gamma}\big)\mathcal{Q}(z)\frac{dz}{z}\Big)\Big(\int_{\mathbb{X}}\big(\ell^{\delta}(x)-\ell(x)\big)\phi_m(x)dx\Big)^2\\
&\le\delta^2\big(1+\overline{C}\big)^2.
\end{aligned}$$

by the fact that $r_k(\xi) \in [0,1]$. Therefore, combined with (16), we obtain

$$\big\|\mathcal{K}(f^{k(\delta)} - f)\big\|_{L^2(\mathbb{X})} \leq \|\mathcal{K}f^{k(\delta),\delta} - \ell^\delta\|_{L^2(\mathbb{X})} + \delta \leq (\tau + (1+\overline{C}))\delta. \tag{19}$$

Moreover, it follows from (18) that

$$\begin{aligned}\|f^k - f\|^2_{\mathbb{H}_0^{2s}(\mathbb{X})} = \sum_{n=1}^{\infty} \xi_m^{2s} r_k^2 \Big(\int_b^{\mathcal{T}} \big(\log \frac{\mathcal{T}}{z}\big)^{\gamma-1} E_{\gamma,\gamma}\big(-\xi_m^\sigma \big(\log \frac{\mathcal{T}}{z}\big)^\gamma\big) \mathcal{Q}(z) \frac{dz}{z}\Big) \\ \times \Big(\int_{\mathbb{X}} \big(f^0(x) - f(x)\big)\phi_m(x)dx\Big)^2 \leq \mathcal{E}^2.\end{aligned}$$

Combined (19) with Theorem 5.3, we have

$$\begin{aligned}\big\|f^{k(\delta)} - f\big\|_{L^2(\mathbb{X})} &\leq \mathcal{C}\big(\alpha,\sigma,\mathcal{Q}_1,s,\mathcal{T},\xi_1\big)\mathcal{E}^{\frac{\sigma}{2s+\sigma}} \big\|\mathcal{K}\big(f^{k(\delta)} - f\big)\big\|^{\frac{2s}{2s+\sigma}} \\ &\leq \mathcal{C}\big(\alpha,\sigma,\mathcal{Q}_1,s,\mathcal{T},\xi_1\big)(\tau+1)^{\frac{2s}{2s+\sigma}} \mathcal{E}^{\frac{\sigma}{2s+\sigma}} \delta^{\frac{2s}{2s+\sigma}}.\end{aligned} \tag{20}$$

Step 2 Considering $\|f^{k,\delta} - f^k\|_{L^2(\mathbb{X})}$

$$\begin{aligned}&\big\|f^{k,\delta} - f^k\big\|^2_{L^2(\mathbb{X})} \\ &= \sum_{n=1}^{\infty} s_k^2 \Big| \int_b^{\mathcal{T}} \big(\log \frac{\mathcal{T}}{z}\big)^{\gamma-1} E_{\gamma,\gamma}\big(-\xi_m^\sigma \big(\log \frac{\mathcal{T}}{z}\big)^\gamma\big) \mathcal{Q}(z) \frac{dz}{z}\Big| \Big(\int_{\mathbb{X}} \big(\ell^\delta(x) - \ell(x)\big)\phi_m(x)dx\Big)^2 \\ &\leq \sigma_k^2 \delta^2 (1+\overline{C})^2.\end{aligned}$$

by Lemma (5.2). Then it follows from Lemma (5.5) that

$$\begin{aligned}(\tau-1)^2\delta^2 \leq &\Big\|Kf^{k(\delta)-1} - \ell\Big\|^2_{L^2(\mathbb{X})} \\ &= \sum_{n=1}^{\infty} \Big| \int_b^{\mathcal{T}} \big(\log \frac{\mathcal{T}}{z}\big)^{\gamma-1} E_{\gamma,\gamma}\big(-\xi_m^\sigma \big(\log \frac{\mathcal{T}}{z}\big)^\gamma\big) \mathcal{Q}(z) \frac{dz}{z}\Big|^2 r^2_{k(\delta)-1} \\ &\quad \times \Big| \int_b^{\mathcal{T}} \big(\log \frac{\mathcal{T}}{z}\big)^{\gamma-1} E_{\gamma,\gamma}\big(-\xi_m^\sigma \big(\log \frac{\mathcal{T}}{z}\big)^\gamma\big) \mathcal{Q}(z) \frac{dz}{z}\Big| \Big(\int_{\mathbb{X}} \big(f^0(x) - f(x)\big)\phi_m(x)dx\Big)^2 \\ &= \sum_{n=1}^{\infty} \Big| \int_b^{\mathcal{T}} \big(\log \frac{\mathcal{T}}{z}\big)^{\gamma-1} E_{\gamma,\gamma}\big(-\xi_m^\sigma \big(\log \frac{\mathcal{T}}{z}\big)^\gamma\big) \mathcal{Q}(z) \frac{dz}{z}\Big|^{\frac{2\sigma+4s}{\sigma}} r^2_{k(\delta)-1} \\ &\quad \times \Big| \int_b^{\mathcal{T}} \big(\log \frac{\mathcal{T}}{z}\big)^{\gamma-1} E_{\gamma,\gamma}\big(-\xi_m^\sigma \big(\log \frac{\mathcal{T}}{z}\big)^\gamma\big) \mathcal{Q}(z) \frac{dz}{z}\Big| \\ &\quad \times \Big| \int_b^{\mathcal{T}} \big(\log \frac{\mathcal{T}}{z}\big)^{\gamma-1} E_{\gamma,\gamma}\big(-\xi_m^\sigma \big(\log \frac{\mathcal{T}}{z}\big)^\gamma\big) \mathcal{Q}(z) \frac{dz}{z}\Big|^{-\frac{4s}{\sigma}} \\ &\quad \times \Big(\int_{\mathbb{X}} \big(f^0(x) - f(x)\big)\phi_m(x)dx\Big)^2.\end{aligned} \tag{21}$$

By Lemmas (2.4) and (21), we can obtain that

$$\begin{aligned}(\tau-1)^2\delta^2 &\le \left\|\mathcal{K}f^{k(\delta)-1}-\ell\right\|^2_{L^2(\mathbb{X})}\\ &\le C\varsigma_{k(\delta)-1}^{-2(\frac{2s}{\sigma}+1)}\sum_{n=1}^{\infty}\xi_m^{2s}\Big(\int_{\mathbb{X}}\big(f^0(x)-f(x)\big)\phi_m(x)dx\Big)^2\\ &\le C^2(s,\sigma)\underline{C}^{-\frac{4s}{\sigma}}\varsigma_{k(\delta)-1}^{-2(\frac{2s}{\sigma}+1)}\left\|f^0-f\right\|^2_{\mathbb{H}_0^{2s}(\mathbb{X})}\\ &\le C^2(s,\sigma)\underline{C}^{-\frac{4s}{\sigma}}\mathcal{E}^2\varsigma_{k(\delta)-1}^{-\frac{2(2s+\sigma)}{\sigma}} \le C\mathcal{E}^2\varsigma_{k(\delta)}^{-\frac{2(2s+\sigma)}{\sigma}},\end{aligned}$$

and since assumption (2) gives

$$\varsigma_k=\varsigma_{k-1}+\alpha_k^{-1}\le\varsigma_{k-1}+c\varsigma_{k-1}=(1+c)\varsigma_{k-1},\ \text{thus}\ (\tau-1)\delta\le C\mathcal{E}\varsigma_{k(\delta)}^{-\frac{2s+\sigma}{\sigma}},$$

which implies that

$$\varsigma_{k(\delta)}\le\Big(\frac{C\mathcal{E}}{\tau-1}\Big)^{\frac{\sigma}{2s+\sigma}}\delta^{-\frac{\sigma}{2s+\sigma}}.$$

It follows that

$$\left\|f^{k(\delta),\delta}-f^k\right\|_{L^2(\mathbb{X})}\le\Big(\frac{C}{\tau-1}\Big)^{\frac{\sigma}{2s+\sigma}}\mathcal{E}^{\frac{\sigma}{2s+\sigma}}\delta^{\frac{2s}{2s+\sigma}}.$$

Added (20), we obtain estimate (17). Then the proof is accomplished. □

References

[1] B. D. Nghia, N. H. Luc, Xiaolan Qin, Yan Wang, On maximal solution to a degenerate parabolic equation involving in time fractional derivative. Electron. J. Appl. Math., 1 (1), 2023.

[2] D. T. T. Xuan, V. T. T. Ha, Recovering solution of the Reverse nonlinear time Fractional diffusion equations with fluctuations data. Electron. J. Appl. Math., 1 (2), 2023.

[3] X. Feng, L. Eldén and C. Fu, A quasi-boundary-value method for the Cauchy problem for elliptic equations with nonhomogeneous Neumann data, J. Inverse Ill-Posed Probl., 18 (6): 617-645, 2010.

[4] X. Feng, P. Li and X. Wang, An inverse random source problem for the time fractional diffusion equation driven by a fractional Brownian motion. Inverse Problems, 36(4): 045008, 30, 2020.

[5] X. Feng, M. Zhao and Z. Qian, A Tikhonov regularization method for solving a backward time-space fractional diffusion problem. J. Comput. Appl. Math., 411: 114236, 20, 2022.

[6] Y. Deng and Z. Liu, Iteration methods on sideways parabolic equations. Inverse Problems, 25(9), 095004, 14, 2005.

[7] K. Fujishiro and Y. Kian, Determination of time dependent factors of coefficients in fractional diffusion equations. Math. Control Relat. Fields, 6(2): 251-269, 2016.

[8] F. Yang and C. Fu, The quasi-reversibility regularization method for identifying the unknown source for time fractional diffusion equation. Appl. Math. Model., 39(5-6): 1500-1512, 2015.

[9] K. Sakamoto and M. Yamamoto, Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems. Journal of Mathematical Analysis and Applications, 382(1): 426-447, 2011.

[10] K. Fujishiro and Y. Kian, Determination of time dependent factors of coefficients in fractional diffusion equations. Math. Control Relat. Fields, 6(2): 251-269, 2016.

[11] T. Wei and J. Wang, A modified quasi-boundary value method for the backward time-fractional diffusion problem. ESAIM Math. Model. Numer. Anal., 48(2): 603-621, 2014.

[12] T. Wei and J. Wang, A modified quasi-boundary value method for an inverse source problem of the time fractional diffusion equation. Appl. Numer. Math., 78: 95-111, 2014.

[13] Yang, F., Cao, Y., & Li, X. X. (2023). Two regularization methods for identifying the source term of Caputo-Hadamard time-fractional diffusion equation. Mathematical Methods in the Applied Sciences, 46(15), 16170-16202.

[14] F. Yang, C.L. Fu, A simplified Tikhonov regularization method for determining the heat source, Appl. Math. Model. 34 (2010), no. 11, 3286–3299.

[15] F.F. Dou, C.L. Fu, F. Yang, Identifying an unknown source term in a heat equation Inverse Probl. Sci. Eng. 17 (2009), no. 7, 901–913.

[16] Zhang, Y., & Feng, X. (2024). A nonstationary iterated quasi-boundary value method for reconstructing the source term in a time-space fractional diffusion equation. Journal of Computational and Applied Mathematics, 440, 115612.
[17] M. Hanke, A. Neubauer and O. Scherzer, A convergence analysis of the Landweber iteration for nonlinear ill-posed problems. Numer. Math., 72 (1): 21-37, 1995.
[18] J. Wang and T. Wei, An iterative method for backward time-fractional diffusion problem. Numer. Numer. Methods Partial Differ. Equ., 30(6): 2029-2041, 2014.