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On Implicit Neutral Tempered ψ-Caputo Fractional Differential Equations with Delay via Densifiability Techniques

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Abstract

This article is a subject about the existence results for a class of tempered ψ-Caputo fractional differential equations. These problems encompassed nonlinear implicit neutral fractional differential equations involving various types of delays, including finite, infinite, and state-dependent delays. The results are based on the concept of the degree of non densifiability (DND) combined with Darbo type fixed point theorem on Banach space using the properties of the phase space. In the last section, we provide some examples to illustrate our obtained results.

Keywords: The tempered ψ-Caputo fractional derivative, implicit neutral problem, existence, degree of nondensifiability, finite delay, infinite delay, sate dependent delay, fixed point.
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1. Introduction

A powerful mathematical tool for simulating complex events with memory effects and long-range interdependence is fractional calculus. In this discipline, the ideas of integration and differentiation are extended beyond integer orders. There has been a discernible surge in fractional calculus research in recent years. Writers have studied a broad range of results, examining various situations and structures of fractional

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differential equations and inclusions. For further exploration of this subject, readers are directed to relevant monographs [3, 10, 24, 1, 2, 4, 48, 50, 51], and relevant publications like [5, 6, 7, 9].

The concept of tempered fractional derivatives extends beyond the conventional framework of fractional derivatives to accommodate functions characterized by exponentially decaying tails. In the pioneering work by Buschman [14], the initial definitions of fractional integration involving weak singular and exponential kernels were introduced. Demonstrated to be a valuable tool, this extension holds particular significance in applications where memory effects are crucial, such as in viscoelastic materials, nonlocal models in physics, and fractional-order control systems [31, 36, 37]. The tempered fractional derivative enables a more precise depiction of underlying dynamics, encompassing both long-range memory and rapidly decaying behaviors. For a more comprehensive exploration of this subject, additional insights can be found in references [32, 34, 38, 42, 43, 45, 46].

Implicit neutral problems are differential equations involving both the dependent variable and its derivatives, arising in fields like biology, physics, and engineering. They pose significant challenges in mathematical analysis and numerical solutions, see the monographs of Hale [27], Hale and Verduyn Lunel [26], Hino et al. [29], and [33]. The incorporation of tempered fractional derivatives in these problems offers a new perspective on their behavior and characteristics.

In [13], the authors established the existence and uniqueness results to the terminal value problem of the following Hilfer-Katugampola type fractional differential equation:

$$
{}^{\delta}D^{\alpha,\beta}_{0^+}\left[y(t) - h(t, y_t)\right] = g(t, y_t); \quad t \in (0, T],\ 0 < T < \infty,
$$

$$
x(T) = \theta \in \mathbb{R},
$$

$$
y(t) = \phi(t), \quad t \in [-\zeta, 0],\ \zeta > 0,
$$

where ${}^{\delta}D^{\alpha,\beta}_{0^+}$, ${}^{\delta}I^{1-\gamma}_{0^+}$ are the Hilfer-Katugampola fractional derivative of order $\alpha \in (0,1)$ and type $\beta \in [0,1]$ and Katugampola fractional integral of order $1-\gamma$ respectively, where $\gamma = \alpha + \beta - \alpha\beta$, $h, g : (0,T] \times C([-\zeta,0],\mathbb{R}) \to \mathbb{R}$ are two given functions and $\phi \in C([-\zeta,0],\mathbb{R})$. For each function y defined on $[-\zeta, 0]$ and for any $t \in (0,T]$, we denote by y_t the element of $C([-\zeta,0],\mathbb{R})$ defined by:

$$
y_t(\tau) = y(t+\tau), \quad \tau \in [-\zeta, 0].
$$

In tier study, the result is based on the Banach contraction principle and Krasnoselskii's fixed point theorem.

Recent advancements in fractional calculus, particularly the integration of non-variable integrals and fractional derivatives with respect to another function, have inspired mathematicians to propose new mathematical models, enabling a more comprehensive description of respective phenomena. Similarly, in a recent development, Almeida [8] utilized the concept of the fractional derivative in the Caputo sense to introduce a novel fractional derivative known as the ψ-Caputo derivative with respect to another function ψ. This derivative serves to generalize a category of fractional derivatives. Medved et al. [38], defined a modification of the ψ-Caputo derivative, termed the tempered ψ–Caputo derivative. Salim et al. [47], introduced a novel definition of the tempered (k,ψ)-fractional operator and established various properties associated with it.

In [44], the authors established existence and uniqueness results to the following tempered (k,ψ)-Hilfer initial value problem with nonlinear implicit fractional differential equation:

$$
\left({}^{TH}_{k}\mathcal{D}^{\vartheta,r,\lambda;\psi}_{a^+}x\right)(t) = f\left(t, x(t), \left({}^{TH}_{k}\mathcal{D}^{\vartheta,r,\lambda;\psi}_{a^+}x\right)(t)\right); \quad t \in (a,b],
$$

$$
\left({}^{T}_{\lambda}\mathfrak{I}^{k(1-\xi),k;\psi}_{a^+}x\right)(a^+) = x_0,
$$

where ${}_{k}^{TH}\mathcal{D}_{a^+}^{\vartheta,r,\lambda;\psi}$, ${}_{\lambda}^{T}\mathfrak{I}_{a^+}^{k(1-\xi),k;\psi}$ are the tempered (k,ψ)-Hilfer fractional derivative of order $\vartheta \in (0,k)$, $r \in [0,1]$ and index $\lambda \in \mathbb{R}$, and tempered (k,ψ)-fractional integral of order $k(1-\xi)$ and index λ respectively, where $\xi = \frac{1}{k}(r(k-\vartheta)+\vartheta)$, $k > 0$, $x_0 \in \mathbb{R}$, $f : [a,b]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is a given appropriate function. In their study, the result is based on the Banach contraction principle.

The concept of the degree of nondensifiability (DND) was initially introduced by Mora and Mira [40]. It stems from a generalization of space-filling curves, specifically the α-dense curves, which were introduced in the early 1980s by Cherruault and Guillez [16]. The development of this concept owes much to Cherruault [17] and Mora [39]. Very recently, García [20] established a novel fixed-point result based on the DND, showcasing its applicability in broader contexts compared to the Darbo fixed-point theorem (DFPT) and its recognized extensions. For a deeper understanding of the utility of DND in investigating the existence of solutions to specific differential or integral equations within certain Banach spaces, we suggest exploring the works [19, 20, 21, 22].

In [18], the authors studied the existence of solutions to a boundary value problem (BVP) containing the ψ-Caputo fractional derivative of the form:

$$\begin{cases} {}^{C}\mathcal{D}_{\theta_1^+}^{\gamma;\psi} y(\theta) = \mathcal{F}(\theta, y(\theta)); \quad \theta \in I := [\theta_1, \theta_2], \\ y(\theta_1) = y(\theta_2) = \delta, \end{cases}$$

where $\gamma \in (1,2]$, ${}^{C}\mathcal{D}_{\theta_1^+}^{\gamma;\psi}$ denotes the ψ-Caputo fractional derivative of order γ, $\mathcal{F} : I \times X \to X$ is a given function, X is a Banach space with norm $\|\cdot\|$, and δ refers to the null vector in the space X. In their study, they applied a version of the Darbo fixed-point theorem with densifiability techniques (DND).

Motivated by the papers mentioned earlier, we investigate the existence result of solutions for the following implicit neutral fractional problem involving tempered ψ-Caputo fractional differential equations with finite delay:

$$ {}_0^C\mathfrak{D}_\eta^{\zeta,\varpi;\psi}\left(\gamma(\eta) - \wp(\eta,\gamma_\eta)\right) = \aleph\left(\eta, \gamma_\eta,\ {}_0^C\mathfrak{D}_\eta^{\zeta,\varpi;\psi}\left(\gamma(\eta) - \wp(\eta,\gamma_\eta)\right)\right); \eta \in \Lambda := [0,\kappa], \tag{1}$$

$$\gamma(\eta) = \chi(\eta); \qquad \eta \in [-\theta, 0], \tag{2}$$

where $0 < \zeta \leq 1$, $\varpi \geq 0$, $0 < \kappa < \infty$, $\theta > 0$, ${}_0^C\mathfrak{D}_t^{\zeta,\varpi;\psi}$ is the tempered ψ-Caputo fractional derivative of order ζ, Ξ is a Banach space with the norm $\|\cdot\|$, $\wp : \Lambda \times \Pi_\theta \to \Xi$, $\aleph : \Lambda \times \Pi_\theta \times \Xi \to \Xi$ are given functions, $\chi \in \Pi_\theta$ and $\Pi_\theta := C([-\theta,0],\Xi)$. For any $\eta \in \Lambda$, we defined $\gamma_\eta \in \Pi_\theta$ by

$$\gamma_\eta(s) = \gamma(\eta + s); \quad \text{for} \quad s \in [-\theta, 0].$$

Next we consider the following implicit neutral fractional problem involving tempered ψ-Caputo fractional differential equations with infinite delay:

$$ {}_0^C\mathfrak{D}_\eta^{\zeta,\varpi;\psi}\left(\gamma(\eta) - \wp(\eta,\gamma_\eta)\right) = \aleph\left(\eta, \gamma_\eta,\ {}_0^C\mathfrak{D}_\eta^{\zeta,\varpi;\psi}\left(\gamma(\eta) - \wp(\eta,\gamma_\eta)\right)\right); \eta \in \Lambda, \tag{3}$$

$$\gamma(\eta) = \chi(\eta); \qquad \eta \in (-\infty, 0], \tag{4}$$

where $\wp : \Lambda \times \mathcal{F} \to \Xi$, $\aleph : \Lambda \times \mathcal{F} \times \Xi \to \Xi$ are given functions, $\chi \in \mathcal{F}$ and $\mathcal{F}$ is the phase space to be specifed later. For any $\eta \in \Lambda$, we defined $\gamma_\eta \in \mathcal{F}$ by:

$$\gamma_\eta(s) = \gamma(\eta + s); \quad \text{for} \quad s \in (-\infty, 0].$$

In addition to that, we study the following problem of implicit neutral fractional problem involving tempered ψ-Caputo fractional differential equations with state-dependent delay (the finite delay case):

$$ {}_0^C\mathfrak{D}_\eta^{\zeta,\varpi;\psi}\left(\gamma(\eta) - \wp(\eta,\gamma_{\varrho(\eta,\gamma_\eta)})\right) = \aleph\left(\eta, \gamma_{\varrho(\eta,\gamma_\eta)},\ {}_0^C\mathfrak{D}_\eta^{\zeta,\varpi;\psi}\left(\gamma(\eta) - \wp(\eta,\gamma_{\varrho(\eta,\gamma_\eta)})\right)\right);\ \eta \in \Lambda, \tag{5}$$

$$\gamma(\eta) = \chi(\eta); \qquad \eta \in [-\theta, 0], \tag{6}$$

where $\varrho : \Lambda \times \Pi_\theta \to \mathbb{R}$, $\wp : \Lambda \times \Pi_\theta \to \Xi$, $\aleph : \Lambda \times \Pi_\theta \times \Xi \to \Xi$ are given functions, $\chi \in \Pi_\theta$.

Finally, we treat the last problem which is an implicit neutral fractional problem involving tempered ψ-Caputo fractional differential equations with state-dependent delay (the infinite delay case):

$${}^{C}_{0}\mathfrak{D}^{\zeta,\varpi;\psi}_{\eta}\left(\gamma(\eta) - \wp(\eta, \gamma_{\varrho(\eta,\gamma_\eta)})\right) = \aleph\left(\eta, \gamma_{\varrho(\eta,\gamma_\eta)}, {}^{C}_{0}\mathfrak{D}^{\zeta,\varpi;\psi}_{\eta}\left(\gamma(\eta) - \wp(\eta, \gamma_{\varrho(\eta,\gamma_\eta)})\right)\right); \ \eta \in \Lambda, \tag{7}$$

$$\gamma(\eta) = \chi(\eta); \qquad \eta \in (-\infty, 0], \tag{8}$$

where $\varrho : \Lambda \times \mathcal{F} \to \mathbb{R}$, $\wp : \Lambda \times \mathcal{F} \to \Xi$, $\aleph : \Lambda \times \mathcal{F} \times \Xi \to \Xi$ are given functions, $\chi \in \Pi_\theta$.

This paper is arranged as follows: Section 2 introduces some preliminaries, definitions, lemmas and auxiliary results that are used throughout this work. In section 3, we give some existence results for the problem (1)-(2) that are based on Darbo's type fixed point theorem with the technique of degree of nondensifiability (DND). The same study applied to problem (1)-(2) in last section, we apply in section 4 to problem (3)-(4) and in section 5 to problems (5)-(6) and (7)-(8). Finally we present some examples to show the validity of our results.

2. Preliminaries

First, we give the definitions and the notations that we will use throughout this paper.

Let $\mathfrak{C} = C(\Lambda, \Xi)$ be the Banach space of all continuous functions γ from Λ into Ξ with the supremum (uniform) norm

$$\|\gamma\|_\infty := \sup_{\eta\in\Lambda} \|\gamma(\eta)\|.$$

Let $C^n(\Lambda, \Xi)$ be the space of continuous functions, n-times continuously differentiable functions from Λ into Ξ.

And $L^\infty(\Lambda, \mathbb{R})$ denotes the Banach space of measurable functions $\gamma : \Lambda \to \mathbb{R}$ which are bounded, equipped with the norm

$$\|\gamma\|_{L^\infty} = \inf\{c > 0 : |\gamma(\eta)| \le c, \quad a.e. \ \eta \in \Lambda\}.$$

Consider $\Pi_\theta := C([-\theta, 0], \Xi)$ the Banach space with the norm

$$\|\gamma\|_{\Pi_\theta} = \sup_{\eta\in[-\theta,0]} \|\gamma(\eta)\|.$$

Let $\mathcal{F}$ be the phase space introduced by Hale and Kato in [25] and follow the terminology used in [30]. Thus, $(\mathcal{F}, \|\cdot\|_{\mathcal{F}})$ will be seminormed linear space of functions mapping $(-\infty, 0]$ into Ξ, and satisfying the following axioms :

(A_1) If $\gamma : (-\infty, \kappa] \to \Xi$ is continuous on Λ and $\gamma_0 \in \mathcal{F}$, then for every $\eta \in \Lambda$ the following conditions hold:

  (i) $\gamma_\eta \in \mathcal{F}$;

  (ii) There exists a positive constant H such that $\|\gamma(\eta)\| \le H\|\gamma_\eta\|_{\mathcal{F}}$;

  (iii) There exist two functions $L(\cdot), M(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ independent of γ with L continuous and bounded and M locally bounded such that

$$\|\gamma_\eta\|_{\mathcal{F}} \le L(\eta) \sup_{s\in[0,\eta]} \|\gamma(s)\| + M(\eta)\|\gamma_0\|_{\mathcal{F}}.$$

(A_2) For the function γ in (A_1), γ_η is a $\mathcal{F}$-valued continuous function on Λ.

(A_3) The space $\mathcal{F}$ is complete.

Denote $L^* = \sup\limits_{\eta \in \Lambda} L(\eta), \quad M^* = \sup\limits_{\eta \in \Lambda} M(\eta).$

Remark 2.1.

1. *$(A_1)(ii)$ is equivalent to $\|\chi(0)\| \leq H\|\chi\|_{\mathcal{F}}$ for every $\chi \in \mathcal{F}$.*
2. *Since $\|\cdot\|_{\mathcal{F}}$ is a seminorm, two elements $\chi, \upsilon \in \mathcal{F}$ can verify $\|\chi - \upsilon\|_{\mathcal{F}} = 0$ without necessarily $\chi(s) = \upsilon(s)$ for all $s \leq 0$.*
3. *From the equivalence in the first remark, we can see that, for all $\chi, \upsilon \in \mathcal{F}$ such that $\|\chi - \upsilon\|_{\mathcal{F}} = 0$. We necessarily have that $\chi(0) = \upsilon(0)$.*

2.1. Tempered ψ-Fractional Calculus

Definition 2.2 (The tempered ψ-fractional integral [38, 35])**.** *Let $\zeta > 0$, $\gamma \in \mathfrak{C}$, $\varpi \geq 0$ and $\psi \in C^1([0,\kappa], \Xi)$ is an increasing differentiable function such that $\psi'(\eta) \neq 0$ for all $\eta \in [0,\kappa]$. Then, the tempered ψ-fractional integral of order ζ is defined by:*

$$\begin{aligned} {}_0I_\eta^{\zeta,\varpi;\psi}\gamma(\eta) &= e^{-\varpi\psi(\eta)}\ {}_0I_\eta^{\zeta;\psi}\left(e^{\varpi\psi(\eta)}\gamma(\eta)\right) \\ &= \int_0^\eta \Psi_\zeta^{\varpi;\psi}(\eta,\tau)\psi'(\tau)\gamma(\tau)d\tau, \end{aligned} \tag{9}$$

where $\Psi_\zeta^{\varpi;\psi}(\eta,\tau) = \frac{e^{-\varpi(\psi(\eta)-\psi(\tau))}[\psi(\eta)-\psi(\tau)]^{\zeta-1}}{\Gamma(\zeta)}$ and ${}_0I_\eta^{\zeta;\psi}$ denotes the ψ-Riemann-Liouville fractional integral [38], defined by:

$${}_0I_\eta^{\zeta;\psi}\gamma(\eta) = \frac{1}{\Gamma(\zeta)}\int_0^\eta \psi'(\tau)[\psi(\eta)-\psi(\tau)]^{\zeta-1}\gamma(\tau)d\tau. \tag{10}$$

Obviously, the tempered ψ-fractional integral (9) reduces to the ψ-Riemann-Liouville fractional integral (10) if $\varpi = 0$.

Definition 2.3 (The tempered ψ-Caputo fractional derivative [38, 35])**.**
Let $\psi \in C^n([0,\kappa], \Xi)$ is an increasing differentiable function such that $\psi'(\eta) \neq 0$ for all $\eta \in [0,\kappa]$, $n - 1 < \zeta < n$; $n \in \mathbb{N}^+$, $\varpi \geq 0$. The tempered ψ-Caputo fractional derivative of order ζ is defined as:

$$\begin{aligned} {}_0^C\mathfrak{D}_\eta^{\zeta,\varpi;\psi}\gamma(\eta) &= e^{-\varpi\psi(\eta)}\ {}_0^C\mathfrak{D}_\eta^{\zeta;\psi}\left(e^{\varpi\psi(\eta)}\gamma(\eta)\right) \\ &= \frac{e^{-\varpi\psi(\eta)}}{\Gamma(n-\zeta)}\int_0^\eta \psi'(\tau)[\psi(\eta)-\psi(\tau)]^{n-\zeta-1}\mathcal{D}_\psi^n\left(e^{\varpi\psi(\tau)}\gamma(\tau)\right)d\tau, \end{aligned}$$

where $\mathcal{D}_\psi^n = \left[\frac{1}{\psi'(\eta)}\frac{d}{d\eta}\right]^n$ and ${}_0^C\mathfrak{D}_\eta^{\zeta,\varpi}$ denotes the ψ-Caputo fractional derivative [38], given by:

$${}_0^C\mathfrak{D}_\eta^{\zeta;\psi}\gamma(\eta) = \frac{1}{\Gamma(n-\zeta)}\int_0^\varrho \psi'(\tau)[\psi(\eta)-\psi(\tau)]^{n-\zeta-1}\mathcal{D}_\psi^n\gamma(\tau)d\tau.$$

Remark 2.4. *If we modify the parameter ϖ, and the function ψ, the tempered ψ-Caputo fractional derivative interpolate the following fractional derivatives:*

- *The Caputo tempered fractional derivative ($\psi(\eta) = \eta$) [34, 49];*
- *The ψ-Caputo fractional derivative ($\varpi = 0$) [11, 12];*
- *The Caputo fractional derivative ($\varpi = 0,\ \psi(\eta) = \eta$) [11, 12];*
- *the Caputo-Hadamard fractional derivative ($\varpi = 0,\ \psi(\eta) = ln\ \eta$) [11, 12].*

Lemma 2.5. *[38] Let $\gamma \in C^n(\Lambda, \Xi)$, $\varpi \geq 0$ and $n-1 < \zeta < n$. Then we have:*

$${}_0I_\eta^{\zeta,\varpi;\psi}\left[{}_0^C\mathfrak{D}_\eta^{\zeta,\varpi;\psi}\gamma(\eta)\right] = \gamma(\eta) - e^{-\varpi\psi(\eta)}\sum_{k=0}^{n-1}\frac{[\psi(\eta)-\psi(0)]^k}{k!}\left[\mathcal{D}_\psi^k\left(e^{\varpi\psi(\eta)}\gamma(\eta)\right)\Big|_{\eta=0}\right].$$

In particular, if $\gamma \in C^1(\Lambda, \Xi)$, then, ${}_0^C\mathfrak{D}_\eta^{\zeta,\varpi;\psi}\left[{}_0I_\eta^{\zeta,\varpi;\psi}\gamma(\eta)\right] = \gamma(\eta)$.

Lemma 2.6. *Let $0 < \zeta < 1$, and $\mathfrak{h} : \Lambda \to \Xi$, $\mathfrak{Q} : \Lambda \to \Xi$ be two continuous functions. Then the problem*

$${}_0^C\mathfrak{D}_\eta^{\zeta,\varpi;\psi}\left(\gamma(\eta) - \mathfrak{h}(\eta)\right) = \mathfrak{Q}(\eta); \quad \eta \in \Lambda, \tag{11}$$

$$\gamma(\eta) = \chi(\eta); \qquad \eta \in \Theta \subseteq \mathbb{R}_-, \tag{12}$$

has a unique solution defined by

$$\gamma(t) = \begin{cases} e^{-\varpi(\psi(\eta)-\psi(0))}[\chi(0) - \mathfrak{h}(0)] + \mathfrak{h}(\eta) + \int_0^\eta \Psi_\zeta^{\varpi;\psi}(\eta,\tau)\psi'(\tau)\mathfrak{Q}(\tau)d\tau; & \eta \in \Lambda, \\ \gamma(\eta) = \chi(\eta); \quad \eta \in \Theta. \end{cases} \tag{13}$$

Proof. Applying The tempered ψ-fractional integral of order ζ on both sides of the equation (11), and by using Lemma 2.5 and if $\eta \in \Lambda$, we get

$$\gamma(\eta) - \mathfrak{h}(\eta) - e^{-\varpi(\psi(\eta)-\psi(0))}[\gamma(0) - \mathfrak{h}(0)] = \int_0^\eta \Psi_\zeta^{\varpi;\psi}(\eta,\tau)\psi'(\tau)\mathfrak{Q}(\tau)d\tau.$$

From the condition (12), we get

$$\gamma(\eta) = e^{-\varpi(\psi(\eta)-\psi(0))}[\chi(0) - \mathfrak{h}(0)] + \mathfrak{h}(\eta) + \int_0^\eta \Psi_\zeta^{\varpi;\psi}(\eta,\tau)\psi'(\tau)\mathfrak{Q}(\tau)d\tau, \quad \eta \in \Lambda.$$

Conversely, we can easily show by Definition 2.3 and Lemma 2.5 that if γ verifies (13), then it satisfies the problem (11)-(12). ☐

2.2. Degree of Nondensifiability

Definition 2.7. *[39, 41] Assume that $\mathcal{B}_\Xi$ is the class of non-empty and bounded subsets of Ξ, $\alpha \geq 0$ and $\mathcal{U} \in \mathcal{B}_\Xi$; a continuous mapping $\sigma : \mathfrak{J} := [0,1] \to \Xi$ is said to be an α-dense curve in $\mathcal{U}$ if the following conditions hold:*

- *$\sigma(\mathfrak{J}) \subset \mathcal{U}$;*
- *For any $\gamma_1 \in \mathcal{U}$, there is $\gamma_2 \in \sigma(\mathfrak{J})$ such that $\|\gamma_1 - \gamma_2\| \leq \alpha$.*

If for each $\alpha > 0$, there is an α-dense curve in $\mathcal{U}$, then $\mathcal{U}$ is said to be densifiable.

Definition 2.8. *[40, 23] Let $\alpha \geq 0$, and denote by $\Delta_{\alpha,\mathcal{U}}$ the class of all α-dense curves in $\mathcal{U} \in \mathcal{B}_\Xi$. The degree of nondensifiability (DND) is a mapping $\vartheta : \mathcal{B}_\Xi \to \mathbb{R}_+$ defined as follows:*

$$\vartheta(\mathcal{U}) = inf\{\alpha \geq 0 : \Delta_{\alpha,\mathcal{U}} \neq \emptyset\},$$

for each $\mathcal{U} \in \mathcal{B}_\Xi$.

Lemma 2.9. *[22, 23] Let $\mathcal{V}$, $\mathcal{U} \in \mathcal{B}_\Xi$, then, the DND satisfies the following properties:*

- *$\vartheta(\mathcal{V}) = 0 \Leftrightarrow \mathcal{V}$ is a precompact set, for each nonempty, bounded, and arc-connected subset $\mathcal{V}$ of Ξ;*
- *$\vartheta(\overline{\mathcal{V}}) = \vartheta(\mathcal{V})$, where $\overline{\mathcal{V}}$ denotes the closure of $\mathcal{V}$;*

- $\vartheta(\mathcal{V}+\mathcal{U}) \leq \vartheta(\mathcal{V})+\vartheta(\mathcal{U})$;
- $\vartheta(\lambda\mathcal{V}) = |\lambda|\vartheta(\mathcal{V})$ *for* $\lambda \in \mathbb{R}$*;*
- $\vartheta(conv(\mathcal{V})) \leq \vartheta(\mathcal{V})$, *where* $conv(\mathcal{V})$ *represent the convex hull of* $\mathcal{V}$*;*
- $\vartheta(conv(\mathcal{V}\cup\mathcal{U})) \leq max\{\vartheta(conv(\mathcal{V})), \vartheta(conv(\mathcal{U}))\}$;
- $\vartheta(\gamma+\mathcal{V}) = \vartheta(\mathcal{V})$ *for all* $\gamma \in \Xi$.

Lemma 2.10. *[20] Let* $\mathcal{U} \subset \mathfrak{C}$ *be non-empty and bounded. Then*

$$\sup_{\eta\in\Lambda} \vartheta(\mathcal{U}(\eta)) \leq \vartheta(\mathcal{U}).$$

For our purpose we will need the following version of the Darbo fixed-point theorem for the DND :

Theorem 2.11. *[20] Let* $\mathfrak{K} \subset \mathcal{B}_\Xi$ *convex and closed and* $\mathbb{Z} : \mathfrak{K} \to \mathfrak{K}$ *continuous. Assume that there is* $0 < \lambda < 1$ *such that*

$$\vartheta(\mathbb{Z}(\mathcal{U})) \leq \lambda\vartheta(\mathcal{U}), \tag{14}$$

for each non-empty and convex $\mathcal{U} \subset \mathfrak{K}$*. Then,* $\mathbb{Z}$ *has some fixed point.*

3. Implicit Neutral Tempered ψ-Caputo Fractional Differential Equations with finite Delay

In this section, we give our main existence result for the problem (1)-(2). Let

$$\Pi = \{\gamma : [-\theta,\kappa] \to \Xi;\ \gamma\mid_{[-\theta,0]} \in \Pi_\theta,\ \gamma\mid_{[0,\kappa]} \in \mathfrak{C}\}.$$

We note that Π is Banach space with the norm

$$\|\gamma\|_\Pi = \sup_{\eta\in[-\theta,\kappa]} \|\gamma(\eta)\|.$$

Definition 3.1. *By a solution of the problem (1)-(2), we mean a function* $\gamma \in \Pi$ *that satisfies the equation (1) on* Λ*, and the initial condition (2) on* $[-\theta,0]$.

As a consequence of Lemma 2.6, we give the following result.

Lemma 3.2. *Let* $\aleph : \Lambda \times \Pi_\theta \times \Xi \to \Xi$*,* $\wp : \Lambda \times \Pi_\theta \to \Xi$ *be two continuous functions. Then the problem (1)-(2) is equivalent to the following integral equation*

$$\gamma(\eta) = \begin{cases} e^{-\varpi(\psi(\eta)-\psi(0))}[\chi(0)-\wp(0,\chi)] + \wp(\eta,\gamma_\eta) \\ \\ +\int_0^\eta \Psi_\zeta^{\varpi;\psi}(\eta,\tau)\psi'(\tau)\aleph(\tau,\gamma_\tau,\mathfrak{f}(\tau))d\tau; \quad \eta\in\Lambda, \\ \\ \gamma(\eta) = \chi(\eta); \quad \eta\in[-\theta,0], \end{cases} \tag{15}$$

where $\mathfrak{f} \in \mathfrak{C}$ *satisfies the following functional equation*

$$\mathfrak{f}(\eta) = \aleph(\eta,\gamma_\eta,\mathfrak{f}(\eta)).$$

The following hypotheses we will use in the sequel :

(H_1) $\aleph : \Lambda \times \Pi_\theta \times \Xi \to \Xi$ is a Carathéodory function.

(H_2) For each $\eta \in \Lambda$, the function $\wp(\eta,\cdot)$ is continuous.

(H_3) There exist two functions $\mu, \nu \in L^{\infty}(\Lambda, \mathbb{R}^+)$ and two continuous nondecreasing functions $\varphi_1, \varphi_2 : [0, \infty) \to [0, \infty)$ such that:

$$\|\aleph(\eta, \gamma_1, \gamma_2)\| \leq \mu(\eta)\varphi_1(\|\gamma_1\|_{\Pi_\theta}) + \nu(\eta)\varphi_2(\|\gamma_2\|) \quad \text{for } \gamma_1 \in \Pi_\theta, \gamma_2 \in \Xi,$$

where $\|\nu\|_{L^\infty}\|\varphi_2\|_\infty < 1$.

(H_4) There exists a function $\xi \in L^{\infty}(\Lambda, \mathbb{R}^+)$ and a continuous nondecreasing function $\varphi_3 : [0, \infty) \to [0, \infty)$, such that:

$$\|\wp(\eta, \gamma)\| \leq \xi(\eta)\varphi_3(\|\gamma\|_{\Pi_\theta}) \quad \text{for } \gamma \in \Pi_\theta.$$

(H_5) There exist two functions $\beta_1, \beta_2 \in L^{\infty}(\Lambda, \mathbb{R}^+)$ where $\|\beta_2\|_{L^\infty} < 1$ such that for any non-empty, bounded, and convex $\mathcal{V} \in \Xi$ and $\mathcal{U} \in \Pi_\theta$ the following inequality holds for each $\eta \in \Lambda$:

$$\vartheta(\aleph(\eta, \mathcal{U}, \mathcal{V})) \leq \beta_1(\eta) \sup_{s \in [-\theta, 0]} \vartheta\left(\mathcal{U}(s)\right) + \beta_2(\eta)\vartheta(\mathcal{V}).$$

(H_6) There exists a function $\beta_3 \in L^{\infty}(\Lambda, \mathbb{R}^+)$ such that for any non-empty, bounded, and convex $\mathcal{U} \in \Pi_\theta$ the following inequality hold for each $\eta \in \Lambda$:

$$\vartheta(\wp(\eta, \mathcal{U})) \leq \beta_3(\eta) \sup_{s \in [-\theta, 0]} \vartheta\left(\mathcal{U}(s)\right).$$

(H_7) There exists a constant $\mathcal{R} > 0$ such that

$$\|\chi\|_{\Pi_\theta} + \|\xi\|_{L^\infty} \left[\varphi_3(\|\chi\|_{\Pi_\theta}) + \varphi_3(\mathcal{R})\right] + \frac{[\psi(\kappa) - \psi(0)]^\zeta \varphi(\mathcal{R})}{\Gamma(\zeta + 1)} \leq \mathcal{R}, \tag{16}$$

where φ is the comparison function defined by

$$\varphi(\eta) = (Id - \|\nu\|_{L^\infty}\varphi_2)^{-1}(\|\mu\|_{L^\infty}\varphi_1(\eta)),$$

where Id is the identity function.

Theorem 3.3. *Assume that the hypotheses $(H_1) - (H_7)$, and the condition*

$$0 < \frac{\|\beta_3\|_{L^\infty} + \|\beta_1\|_{L^\infty}[\psi(\kappa) - \psi(0)]^\zeta}{(1 - \|\beta_2\|_{L^\infty})\Gamma(\zeta + 1)} < 1, \tag{17}$$

hold. Then the problem (1)-(2) has at least one solution defined on $[-\theta, \kappa]$.

Proof. Transform the problem (1)-(2) into a fixed point equation. Consider the operator $\mathbb{Z} : \mathfrak{C} \to \mathfrak{C}$ defined by:

$$(\mathbb{Z}\gamma)(\eta) = \begin{cases} e^{-\varpi(\psi(\eta) - \psi(0))}[\chi(0) - \wp(0, \chi)] + \wp(\eta, \gamma_\eta) \\ \\ + \int_0^\eta \Psi_\zeta^{\varpi;\psi}(\eta, \tau)\psi'(\tau)\aleph(\tau, \gamma_\tau, \mathfrak{f}(\tau))d\tau; \quad \eta \in \Lambda, \\ \\ \gamma(\eta) = \chi(\eta); \quad \eta \in [-\theta, 0], \end{cases} \tag{18}$$

where $\mathfrak{f} \in \mathfrak{C}$ satisfies the following functional equation

$$\mathfrak{f}(\eta) = \aleph(\eta, \gamma_\eta, \mathfrak{f}(\eta)).$$

Let

$$\Theta = \{\gamma \in \Pi, \ \|\gamma\|_\Pi \leq \mathcal{R}\}, \tag{19}$$

where $\mathcal{R}$ satisfies the inequality (16).

Clearly, the subset Θ is bounded, closed and convex. We shall show that the operator $\mathbb{Z}$ satisfies all the assumptions of Theorem 2.11. The proof will be given in three steps.

Step 1. *$\mathbb{Z}$ maps Θ into itself.*
Let $\gamma \in \Theta$. Then for each $\eta \in [-\theta, 0]$, we have

$$\|(\mathbb{Z}\gamma)(\eta)\| = \|\chi(\eta)\| \leq \|\chi\|_{\Pi_\theta} \leq \mathcal{R},$$

and for each $\eta \in \Lambda$, we get

$$\begin{aligned}\|(\mathbb{Z}\gamma)(\eta)\| &\leq e^{-\varpi(\psi(\eta)-\psi(0))}\|\chi(0) - \wp(0,\chi)\| + \|\wp(\eta,\gamma_\eta)\| \\ &+ \int_0^\eta \Psi_\zeta^{\varpi;\psi}(\eta,\tau)\psi'(\tau)\|\aleph(\tau,\gamma_\tau,\mathfrak{f}(\tau))\|d\tau \\ &\leq \|\chi(0)\| + \|\wp(0,\chi)\| + \|\wp(\eta,\gamma_\eta)\| \\ &+ \int_0^\eta \frac{[\psi(\eta)-\psi(\tau)]^{\zeta-1}}{\Gamma(\zeta)}\psi'(\tau)\|\aleph(\tau,\gamma_\tau,\mathfrak{f}(\tau))\|d\tau,\end{aligned}$$

where $\mathfrak{f} \in \mathfrak{C}$ with $\mathfrak{f}(\eta) = \aleph(\eta,\gamma_\eta,\mathfrak{f}(\eta))$. From (H_3), for each $\eta \in \Lambda$, we get

$$\begin{aligned}\|\mathfrak{f}(\eta)\| &= \|\aleph(\eta,\gamma_\eta,\mathfrak{f}(\eta))\| \\ &\leq \mu(\eta)\varphi_1(\|\gamma_\eta\|_{\Pi_\theta}) + \nu(\eta)\varphi_2(\|\mathfrak{f}(\eta)\|) \\ &\leq \|\mu\|_{L^\infty}\varphi_1(\|\gamma\|_\Pi) + \|\nu\|_{L^\infty}\varphi_2(\|\mathfrak{f}(\eta)\|) \\ &\leq \|\mu\|_{L^\infty}\varphi_1(\mathcal{R}) + \|\nu\|_{L^\infty}\varphi_2(\|\mathfrak{f}(\eta)\|).\end{aligned}$$

This gives

$$\|\mathfrak{f}(\eta)\| \leq (Id - \|\nu\|_{L^\infty}\varphi_2)^{-1}(\|\mu\|_{L^\infty}\varphi_1(\mathcal{R})) = \varphi(\mathcal{R}). \tag{20}$$

Thus, by using (H_4) and (20), we get

$$\begin{aligned}\|(\mathbb{Z}\gamma)(\eta)\| &\leq \|\chi\|_{\Pi_\theta} + \xi(\eta)\varphi_3(\|\chi\|_{\Pi_\theta}) + \xi(\eta)\varphi_3(\|\gamma_\eta\|_{\Pi_\theta}) \\ &+ \int_0^\eta \frac{[\psi(\eta)-\psi(\tau)]^{\zeta-1}}{\Gamma(\zeta)}\psi'(\tau)\varphi(\mathcal{R})d\tau \\ &\leq \|\chi\|_{\Pi_\theta} + \|\xi\|_{L^\infty}\varphi_3(\|\chi\|_{\Pi_\theta}) + \|\xi\|_{L^\infty}\varphi_3(\|\gamma\|_\Pi) + \frac{[\psi(\eta)-\psi(0)]^\zeta\varphi(\mathcal{R})}{\Gamma(\zeta+1)} \\ &\leq \|\chi\|_{\Pi_\theta} + \|\xi\|_{L^\infty}\left[\varphi_3(\|\chi\|_{\Pi_\theta}) + \varphi_3(\mathcal{R})\right] + \frac{[\psi(\kappa)-\psi(0)]^\zeta\varphi(\mathcal{R})}{\Gamma(\zeta+1)}.\end{aligned}$$

From (H_7), we get $\|(\mathbb{Z}\xi)(\eta)\|_\Pi \leq \mathcal{R}$.
Hence $\mathbb{Z}(\Theta) \subset \Theta$.

Step 2. *$\mathbb{Z}$ is continuous.*
Let (γ^n) be a sequence such that $\gamma^n \to \gamma$ in Θ. If $\eta \in [-\theta, 0]$, we have

$$\|\mathbb{Z}\gamma^n(\eta) - \mathbb{Z}\gamma(\eta)\| = 0.$$

If $\eta \in \Lambda$, Since $\aleph$ and $\wp$ satisfie assumptions (H_1) and (H_2) respectively, we have $\aleph(\eta,\gamma^n_\eta,\mathfrak{f}^n(\eta))$ and $\wp(\eta,\gamma^n_\eta)$ converging uniformly to $\aleph(\eta,\gamma_\eta,\mathfrak{f}(\eta))$ and $\wp(\eta,\gamma_\eta)$ respectively. Hence the Lebesgue dominated convergence theorem implies $(\mathbb{Z}\gamma^n)(\eta)$ converging uniformly to $(\mathbb{Z}\gamma)(\eta)$ in Θ, for each $\eta \in \Lambda$. Hence $\mathbb{Z}\gamma^n \to \mathbb{Z}\gamma$. Then $\mathbb{Z} : \Theta \to \Theta$ is continuous.

Step 3. *The condition (14) holds.*
Let $\mathcal{U}$ be any non-empty and convex subset of Θ. If $\eta \in [-\theta, 0]$, then

$$\vartheta(\mathbb{Z}(\mathcal{U})) = 0,$$

where $\mathcal{U} = \{\chi(\eta); \quad \eta \in [-\theta, 0]\}$.
If $\eta \in \Lambda$, from (H_5), we have

$$\vartheta(\aleph(\eta,\mathcal{U}_\eta,\mathcal{V}(\eta))) \leq \beta_1(\eta) \sup_{s\in[-\theta,0]} \vartheta\left(\mathcal{U}_\eta(s)\right) + \beta_2(\eta)\vartheta(\mathcal{V}(\eta)); \quad \text{for a.e} \quad \eta \in \Lambda,$$

with $\mathcal{V}(\eta) = \aleph(\eta,\mathcal{U}_\eta,\mathcal{V}(\eta))$. Then

$$\begin{aligned}\vartheta(\mathcal{V}(\eta)) &= \vartheta(\aleph(\eta,\mathcal{U}_\eta,\mathcal{V}(\eta)))\\ &\leq \beta_1(\eta) \sup_{s\in[-\theta,0]} \vartheta\left(\mathcal{U}_\eta(s)\right) + \beta_2(\eta)\vartheta(\mathcal{V}(\eta))\\ &\leq \|\beta_1\|_{L^\infty} \sup_{s\in[-\theta,0]} \vartheta\left(\mathcal{U}_\eta(s)\right) + \|\beta_2\|_{L^\infty}\vartheta(\mathcal{V}(\eta)).\end{aligned}$$

This gives

$$\vartheta(\mathcal{V}(\eta)) \leq \frac{\|\beta_1\|_{L^\infty}}{1-\|\beta_2\|_{L^\infty}} \sup_{s\in[-\theta,0]} \vartheta\left(\mathcal{U}_\eta(s)\right).$$

Therefore, given any $\epsilon > 0$, there is a continuous mapping $\sigma_\eta : \mathfrak{J} \to \Xi$, with $\sigma_\eta(\mathfrak{J}) \subset \vartheta(\aleph(\eta,\gamma_\eta,\mathfrak{f}(\eta)))$, such that for all $\gamma \in \mathcal{U}$ there is $\delta \in \mathfrak{J}$ with:

$$\|\aleph(\eta,\gamma_\eta,\mathfrak{f}(\eta)) - \sigma_\eta(\delta)\| \leq \frac{\|\beta_1\|_{L^\infty}}{1-\|\beta_2\|_{L^\infty}} \sup_{s\in[-\theta,0]} \vartheta\left(\mathcal{U}_\eta(s)\right) + \epsilon; \quad \text{for a.e} \quad \eta \in \Lambda. \tag{21}$$

And, from (H_6), we have

$$\vartheta(\wp(\eta,\mathcal{U}_\eta)) \leq \beta_3(\eta) \sup_{s\in[-\theta,0]} \vartheta\left(\mathcal{U}_\eta(s)\right) \leq \|\beta_3\|_\infty \sup_{s\in[-\theta,0]} \vartheta\left(\mathcal{U}_\eta(s)\right); \quad \text{for a.e} \quad \eta \in \Lambda.$$

Therefore, given any $\epsilon' > 0$, there is a continuous mapping $\rho_\eta : \mathfrak{J} \to \Xi$, with $\rho_\eta(\mathfrak{J}) \subset \vartheta(\wp(\eta,\gamma_\eta))$, such that for all $\gamma \in \mathcal{U}$ there is $\delta \in \mathfrak{J}$ with:

$$\|\wp(\eta,\gamma_\eta) - \rho_\eta(\delta)\| \leq \|\beta_3\|_\infty \sup_{s\in[-\theta,0]} \vartheta\left(\mathcal{U}_\eta(s)\right) + \varepsilon'; \quad \text{for a.e} \quad \eta \in \Lambda. \tag{22}$$

Now, we consider the mapping $\mathbb{M} : \mathfrak{J} \to \mathcal{C}$ defined by:

$$\begin{aligned}\delta \in \mathfrak{J} \longmapsto \mathbb{M}(\delta,\eta) &= e^{-\varpi(\psi(\eta)-\psi(0))}[\chi(0) - \wp(0,\chi)] + \rho_\eta(\delta)\\ &\quad + \int_0^\eta \Psi_\zeta^{\varpi;\psi}(\eta,\tau)\psi'(\tau)\sigma_\tau(\delta)d\tau; \quad \eta \in \Lambda.\end{aligned}$$

It is clear that $\mathbb{M}$ is continuous and $\mathbb{M}(\mathfrak{J}) \subset \mathbb{Z}(\mathcal{U})$. Additionally, by (21) and (22), given $\gamma \in \mathcal{U}$, we can find $\delta \in \mathfrak{J}$ such that:

$$\begin{aligned}\|(\mathbb{Z}\gamma)(\eta) - \mathbb{M}(\delta,\eta)\| &\leq \|\wp(\eta,\gamma_\eta) - \rho_\eta(\delta)\|\\ &\quad + \int_0^\eta \Psi_\zeta^{\varpi;\psi}(\eta,\tau)\psi'(\tau)\|\aleph(\tau,\gamma_\tau,\mathfrak{f}(\tau)) - \sigma_\tau(\delta)\|d\tau\\ &\leq \|\beta_3\|_\infty \sup_{s\in[-\theta,0]} \vartheta\left(\mathcal{U}_\eta(s)\right) + \varepsilon' + \int_0^\eta \frac{[\psi(\eta)-\psi(\tau)]^{\zeta-1}}{\Gamma(\zeta)}\psi'(\tau)\\ &\quad \times \left(\frac{\|\beta_1\|_{L^\infty}}{1-\|\beta_2\|_{L^\infty}} \sup_{s\in[-\theta,0]} \vartheta\left(\mathcal{U}_\eta(s)\right) + \epsilon\right) d\tau.\end{aligned}$$

Using Lemma 2.10 and the features of ϑ, we obtain

$$\begin{aligned}\|(\mathbb{Z}\gamma)(\eta)-\mathbb{M}(\delta,\eta)\| \leq & \|\beta_3\|_{L^\infty}\vartheta(\mathcal{U})+\varepsilon' \\ & +\int_0^\eta \frac{[\psi(\eta)-\psi(\tau)]^{\zeta-1}}{\Gamma(\zeta)}\psi'(\tau)\left(\frac{\|\beta_1\|_{L^\infty}}{1-\|\beta_2\|_{L^\infty}}\vartheta(\mathcal{U})+\epsilon\right)d\tau.\end{aligned}$$

For every $\epsilon>0$ and $\epsilon'>0$, we conclude

$$\|(\mathbb{Z}\gamma)(\eta)-\mathbb{M}(\delta,\eta)\| \leq \frac{\|\beta_3\|_{L^\infty}+\|\beta_1\|_{L^\infty}[\psi(\kappa)-\psi(0)]^\zeta}{(1-\|\beta_2\|_{L^\infty})\Gamma(\zeta+1)}\vartheta(\mathcal{U}).$$

Hence $\vartheta(\mathbb{Z}\mathcal{U})\leq\lambda\vartheta(\mathcal{U})$, with $\lambda=\frac{\|\beta_3\|_{L^\infty}+\|\beta_1\|_{L^\infty}[\psi(\kappa)-\psi(0)]^\zeta}{(1-\|\beta_2\|_{L^\infty})\Gamma(\zeta+1)}$.

As a consequence of the three above steps, from Theorem 2.11, the operator $\mathbb{Z}$ has at least fixed point which is a solution of the problem (1)–(2). ☐

4. Implicit Neutral Tempered ψ-Caputo Fractional Differential Equations with Infinite Delay

In this section, we give our main existence result for the problem (3)-(4). Let

$$\Pi^*=\{\gamma:(-\infty,\kappa]\to\Xi;\ \gamma\,|_{(-\infty,0]}\in\mathcal{F},\ \gamma\,|_{[0,\kappa]}\in\mathfrak{C}\}.$$

We note that Π^* is Banach space with the norm

$$\|\gamma\|_{\Pi^*}=\sup_{\eta\in(-\infty,\kappa]}\|\gamma(\eta)\|.$$

Definition 4.1. *By a solution of the problem (2)-(3), we mean a function $\gamma\in\Pi^*$ that satisfies the equation (2) on Λ, and the initial condition (3) on $(-\infty,0]$.*

As a consequence of Lemma 2.6, we give the following result.

Lemma 4.2. *Let $\aleph:\Lambda\times\mathcal{F}\times\Xi\to\Xi$, $\wp:\Lambda\times\mathcal{F}\to\Xi$ be two continuous functions. Then the problem (3)-(4) is equivalent to the following integral equation*

$$\gamma(\eta)=\begin{cases} e^{-\varpi(\psi(\eta)-\psi(0))}[\chi(0)-\wp(0,\chi)]+\wp(\eta,\gamma_\eta) \\ +\int_0^\eta \Psi_\zeta^{\varpi;\psi}(\eta,\tau)\psi'(\tau)\aleph(\tau,\gamma_\tau,\mathfrak{f}(\tau))d\tau; \quad \eta\in\Lambda, \\ \gamma(\eta)=\chi(\eta); \quad \eta\in(-\infty,0], \end{cases} \tag{23}$$

where $\mathfrak{f}\in\mathfrak{C}$ satisfies the following functional equation

$$\mathfrak{f}(\eta)=\aleph(\eta,\gamma_\eta,\mathfrak{f}(\eta)).$$

The following hypotheses we will use in the sequel :

(h_1) $\aleph:\Lambda\times\mathcal{F}\times\Xi\to\Xi$ is a Carathéodory function.

(h_2) For each $\eta\in\Lambda$, the function $\wp(\eta,\cdot)$ is continuous.

(h_3) There exist two functions $\tilde{\mu},\tilde{\nu}\in L^\infty(\Lambda,\mathbb{R}^+)$ and two continuous nondecreasing functions $\phi_1,\phi_2:[0,\infty)\to[0,\infty)$ such that:

$$\|\aleph(\eta,\gamma_1,\gamma_2)\|\leq\tilde{\mu}(\eta)\phi_1(\|\gamma_1\|_{\mathcal{F}})+\tilde{\nu}(\eta)\phi_2(\|\gamma_2\|) \quad \text{for } \gamma_1\in\mathcal{F},\gamma_2\in\Xi,$$

where $\|\tilde{\nu}\|_{L^\infty}\|\varphi_2\|_\infty<1$.

(h_4) There exists a function $\tilde{\xi} \in L^\infty(\Lambda, \mathbb{R}^+)$ and a continuous nondecreasing function $\phi_3 : [0,\infty) \to [0,\infty)$, such that:
$$\|\wp(\eta,\gamma)\| \leq \tilde{\xi}(\eta)\phi_3(\|\gamma\|_{\mathcal{F}}) \quad \text{for } \gamma \in \mathcal{F}.$$

(h_5) There exist two functions $\tilde{\beta}_1, \tilde{\beta}_2 \in L^\infty(\Lambda, \mathbb{R}^+)$ where $\|\tilde{\beta}_2\|_{L^\infty} < 1$ such that for any non-empty, bounded, and convex $\mathcal{V} \in \Xi$ and $\mathcal{U} \in \mathcal{F}$ the following inequality holds for each $\eta \in \Lambda$:
$$\vartheta(\aleph(\eta,\mathcal{U},\mathcal{V})) \leq \tilde{\beta}_1(\eta) \sup_{s\in(-\infty,0]} \vartheta\left(\mathcal{U}(s)\right) + \tilde{\beta}_2(\eta)\vartheta(\mathcal{V}).$$

(h_6) There exists a function $\tilde{\beta}_3 \in L^\infty(\Lambda, \mathbb{R}^+)$ such that for any non-empty, bounded, and convex $\mathcal{U} \in \mathcal{F}$ the following inequality hold for each $\eta \in \Lambda$:
$$\vartheta(\wp(\eta,\mathcal{U})) \leq \tilde{\beta}_3(\eta) \sup_{s\in(-\infty,0]} \vartheta\left(\mathcal{U}(s)\right).$$

(h_7) There exists a constant $\mathcal{R} > 0$ such that
$$\|\tilde{\xi}\|_{L^\infty}\phi_3(\Delta + L^*\mathcal{R}^*) + \frac{[\psi(\kappa)-\psi(0)]^\zeta\phi(\mathcal{R}^*)}{\Gamma(\zeta+1)} \leq \mathcal{R}^*, \tag{24}$$
where $\Delta = \|\chi\|_{\mathcal{F}}(HL^* + M^*) + L^*\|\tilde{\xi}\|_{L^\infty}\phi_3(\|\chi\|_{\mathcal{F}})$, and ϕ is the comparison function defined by
$$\phi(\eta) = (Id - \|\tilde{\nu}\|_{L^\infty}\phi_2)^{-1}(\|\tilde{\mu}\|_{L^\infty}\phi_1(\Delta + L^*\eta)).$$

Theorem 4.3. *Assume that the hypotheses* $(h_1) - (h_7)$*, and the condition*
$$0 < \frac{\|\tilde{\beta}_3\|_{L^\infty} + \|\tilde{\beta}_1\|_{L^\infty}[\psi(\kappa)-\psi(0)]^\zeta}{(1-\|\tilde{\beta}_2\|_{L^\infty})\Gamma(\zeta+1)} < 1, \tag{25}$$
hold. Then the problem (3)-(4) has at least one solution defined on $(-\infty,\kappa]$*.*

Proof. Transform the problem (3)-(4) into a fixed point equation. Consider the operator $\mathbb{T} : \mathfrak{C} \to \mathfrak{C}$ defined by:
$$(\mathbb{T}\gamma)(\eta) = \begin{cases} e^{-\varpi(\psi(\eta)-\psi(0))}[\chi(0) - \wp(0,\chi)] + \wp(\eta,\gamma_\eta) \\ \\ +\int_0^\eta \Psi_\zeta^{\varpi;\psi}(\eta,\tau)\psi'(\tau)\aleph(\tau,\gamma_\tau,\mathfrak{f}(\tau))d\tau; \quad \eta\in\Lambda, \\ \\ \gamma(\eta) = \chi(\eta); \quad \eta \in (-\infty,0], \end{cases} \tag{26}$$
where $\mathfrak{f} \in \mathfrak{C}$ satisfies the following functional equation
$$\mathfrak{f}(\eta) = \aleph(\eta,\gamma_\eta,\mathfrak{f}(\eta)).$$

Let $x : (-\infty,\kappa] \to \Xi$ be a function defined by
$$x(\eta) = \begin{cases} e^{-\varpi(\psi(\eta)-\psi(0))}[\chi(0) - \wp(0,\chi)], & \eta \in \Lambda, \\ \\ \chi(\eta), & \eta \in (-\infty,0]. \end{cases} \tag{27}$$

Then $x_0 = \chi$ for all $\eta \in (-\infty,0]$. For each $y \in \Pi^*$ with $y(0)=0$, and for each $\eta \in (-\infty,\kappa]$, we denote by $\bar{y}$ the function defined by
$$\bar{y}(t) = \begin{cases} y(t), & \eta \in \Lambda, \\ \\ 0, & \eta \in (-\infty,0]. \end{cases} \tag{28}$$

If γ satisfies $\gamma(\eta) = (\mathbb{T}\gamma)(\eta)$, we can decompose it as $\gamma(\eta) = x(\eta) + y(\eta)$ for $\eta \in \Lambda$, which implies that $\gamma_\eta = x_\eta + y_\eta$ for every $\eta \in \Lambda$, and the function $y(\cdot)$ satisfies

$$y(\eta) = \wp(\eta, x_\eta + y_\eta) + \int_0^\eta \Psi_\zeta^{\varpi;\psi}(\eta,\tau)\psi'(\tau)\aleph(\tau, x_\tau + y_\tau, \mathfrak{f}(\tau))d\tau.$$

Set

$$\Pi_0 = \{y \in \Pi^*, \quad y(0) = 0\},$$

and let $\|\cdot\|_{\Pi_0}$ be the norm in Π_0 defined by

$$\|y\|_{\Pi_0} = \|y_0\|_{\mathcal{F}} + \sup_{\eta\in\Lambda}\|y(\eta)\| = \sup_{\eta\in\Lambda}\|y(\eta)\|; \quad y \in \Pi_0,$$

where Π_0 is a Banach space with the norm $\|\cdot\|_{\Pi_0}$. Defined the operators $\mathcal{P} : \Pi_0 \to \Pi_0$ by

$$\mathcal{P}y(\eta) = \wp(\eta, x_\eta + y_\eta) + \int_0^\eta \Psi_\zeta^{\varpi;\psi}(\eta,\tau)\psi'(\tau)\aleph(\tau, x_\tau + y_\tau, \mathfrak{f}(\tau))d\tau.$$

Obviously, the operator $\mathbb{T}$ has a fixed point is equivalent to $\mathcal{P}$ having a fixed point, and so we turn to proving that $\mathcal{P}$ has a fixed point. Let

$$\Theta^* = \{y \in \Pi_0, \ \|y\|_{\Pi_0} \leq \mathcal{R}^*\}, \tag{29}$$

where $\mathcal{R}^*$ satisfies the inequality (24).

Clearly, the subset Θ^* is bounded, closed and convex. We shall show that the operator $\mathcal{P}$ satisfies all the assumptions of Theorem 2.11. The proof will be given in three steps.

Step 1. *$\mathcal{P}$ maps Θ^* into itself.*
Let $\gamma \in \Theta^*$. Then for each $\eta \in \Lambda$ and from $(A_1) - (A_2)$, it follows that

$$\begin{aligned}
\|x_\eta + y_\gamma\|_{\mathcal{F}} &\leq \|x_\eta\|_{\mathcal{F}} + \|y_\gamma\|_{\mathcal{F}} \\
&\leq L(\eta)\|x(\eta)\| + M(\eta)\|\chi\|_{\mathcal{F}} + L(\eta)\|y(\eta)\| \\
&\leq L^*(\|\chi(0)\| + \|\wp(0,\chi)\|) + M^*\|\chi\|_{\mathcal{F}} + L^*\|y\|_{\Pi_0} \\
&\leq L^*(H\|\chi\|_{\mathcal{F}} + \tilde{\xi}(\eta)\phi_3(\|\chi\|_{\mathcal{F}})) + M^*\|\chi\|_{\mathcal{F}} + L^*\|y\|_{\Pi_0} \\
&\leq \|\chi\|_{\mathcal{F}}(HL^* + M^*) + \|\tilde{\xi}\|_{L^\infty}\phi_3(\|\chi\|_{\mathcal{F}}) + L^*\mathcal{R}^* \\
&= \Delta + L^*\mathcal{R}^*.
\end{aligned}$$

And, from (h_3), we get

$$\begin{aligned}
\|\mathfrak{f}(\eta)\| &= \|\aleph(\eta, x_\eta + y_\eta, \mathfrak{f}(\eta))\| \\
&\leq \tilde{\mu}(\eta)\phi_1(\|x_\eta + y_\eta\|_{\mathcal{F}}) + \tilde{\nu}(\eta)\phi_2(\|\mathfrak{f}(\eta)\|) \\
&\leq \|\tilde{\mu}\|_{L^\infty}\phi_1(\Delta + L^*\mathcal{R}^*) + \|\tilde{\nu}\|_{L^\infty}\phi_2(\|\mathfrak{f}(\eta)\|).
\end{aligned}$$

This gives

$$\|\mathfrak{f}(\eta)\| \leq (Id - \|\tilde{\nu}\|_{L^\infty}\phi_2)^{-1}(\|\tilde{\mu}\|_{L^\infty}\phi_1(\Delta + L^*\mathcal{R}^*)) = \phi(\mathcal{R}). \tag{30}$$

Then, by using (h_4) and (30), we have

$$\begin{aligned}
\|(\mathcal{P}\gamma)(\eta)\| &\leq \|\wp(\eta, x_\eta + y_\eta)\| + \int_0^\eta \Psi_\zeta^{\varpi;\psi}(\eta,\tau)\psi'(\tau)\|\aleph(\tau, x_\eta + y_\eta, \mathfrak{f}(\eta))\|d\tau \\
&\leq \tilde{\xi}(\eta)\phi_3(\|x_\eta + y_\eta\|_{\mathcal{F}}) + \int_0^\eta \frac{[\psi(\varrho) - \psi(\tau)]^{\zeta-1}}{\Gamma(\zeta)}\psi'(\tau)\phi(\mathcal{R}^*)d\tau \\
&\leq \|\tilde{\xi}\|_{L^\infty}\phi_3(\Delta + L^*\mathcal{R}^*) + \frac{[\psi(\kappa) - \psi(0)]^\zeta\phi(\mathcal{R}^*)}{\Gamma(\zeta+1)}.
\end{aligned}$$

From (h_7), we get $\|(\mathcal{P}\xi)(\eta)\|_{\Pi^*} \leq \mathcal{R}^*$.
Hence $\mathcal{P}(\Theta^*) \subset \Theta^*$.

Step 2. *$\mathcal{P}$ is continuous.*
Let (y^n) be a sequence such that $y^n \to y$ in Θ^* for each $\eta \in \Lambda$. Since $\aleph$ and $\wp$ satisfie assumptions (h_1) and (h_2) respectively, we have $\aleph(\eta, x^n_\eta + y^n_\eta, \mathfrak{f}^n(\eta))$ and $\wp(\eta, x^n_\eta + y^n_\eta)$ converging uniformly to $\aleph(\eta, x_\eta + y_\eta, \mathfrak{f}(\eta))$ and $\wp(\eta, x_\eta + y_\eta)$ respectively. Hence the Lebesgue dominated convergence theorem implies $(\mathcal{P}y^n)(\eta)$ converging uniformly to $(\mathcal{P}y)(\eta)$ in Θ^*, for each $\eta \in \Lambda$. Hence $\mathcal{P}y^n \to \mathcal{P}y$. Then $\mathcal{P} : \Theta^* \to \Theta^*$ is continuous.

Step 3. *The condition (14) holds.*
Let $\mathcal{U}$ be any non-empty and convex subset of Θ^*. From (h_5), we have

$$\vartheta(\aleph(\eta,\mathcal{U}_\eta, \mathcal{V}(\eta))) \leq \tilde{\beta}_1(\eta) \sup_{s\in(-\infty,0]} \vartheta\left(\mathcal{U}_\eta(s)\right) + \tilde{\beta}_2(\eta)\vartheta(\mathcal{V}(\eta)); \quad \text{for a.e} \quad \eta \in \Lambda.$$

with $\mathcal{V}(\eta) = \aleph(\eta,\mathcal{U}_\eta, \mathcal{V}(\eta))$. Then

$$\begin{aligned}
\vartheta(\mathcal{V}(\eta)) &= \vartheta(\aleph(\eta,\mathcal{U}_\eta, \mathcal{V}(\eta))) \\
&\leq \tilde{\beta}_1(\eta) \sup_{s\in(-\infty,0]} \vartheta\left(\mathcal{U}_\eta(s)\right) + \tilde{\beta}_2(\eta)\vartheta(\mathcal{V}(\eta)) \\
&\leq \|\tilde{\beta}_1\|_{L^\infty} \sup_{s\in(-\infty,0]} \vartheta\left(\mathcal{U}_\eta(s)\right) + \|\tilde{\beta}_2\|_{L^\infty}\vartheta(\mathcal{V}(\eta)).
\end{aligned}$$

This gives

$$\vartheta(\mathcal{V}(\eta)) \leq \frac{\|\tilde{\beta}_1\|_{L^\infty}}{1 - \|\tilde{\beta}_2\|_{L^\infty}} \sup_{s\in(-\infty,0]} \vartheta\left(\mathcal{U}_\eta(s)\right).$$

Therefore, given any $\epsilon > 0$, there is a continuous mapping $\tilde{\sigma}_\eta : \mathfrak{J} \to \Xi$, with $\tilde{\sigma}_\eta(\mathfrak{J}) \subset \vartheta(\aleph(\eta, x_\eta + y_\eta, \mathfrak{f}(\eta)))$, such that for all $y \in \mathcal{U}$ there is $\delta \in \mathfrak{J}$ with:

$$\|\aleph(\eta, x_\eta + y_\eta, \mathfrak{f}(\eta)) - \tilde{\sigma}_\eta(\delta)\| \leq \frac{\|\tilde{\beta}_1\|_{L^\infty}}{1 - \|\tilde{\beta}_2\|_{L^\infty}} \sup_{s\in(-\infty,0]} \vartheta\left(\mathcal{U}_\eta(s)\right) + \epsilon; \quad \text{for a.e} \quad \eta \in \Lambda. \tag{31}$$

And, from (H_6), we have

$$\vartheta(\wp(\eta,\mathcal{U}_\eta)) \leq \tilde{\beta}_3(\eta) \sup_{s\in(-\infty,0]} \vartheta\left(\mathcal{U}_\eta(s)\right) \leq \|\tilde{\beta}_3\|_\infty \sup_{s\in(-\infty,0]} \vartheta\left(\mathcal{U}_\eta(s)\right); \quad \text{for a.e} \quad \eta \in \Lambda.$$

Therefore, given any $\epsilon' > 0$, there is a continuous mapping $\tilde{\rho}_\eta : \mathfrak{J} \to \Xi$, with $\tilde{\rho}_\eta(\mathfrak{J}) \subset \vartheta(\wp(\eta, x_\eta + y_\eta))$, such that for all $y \in \mathcal{U}$ there is $\delta \in \mathfrak{J}$ with:

$$\|\wp(\eta, x_\eta + y_\eta) - \tilde{\rho}_\eta(\delta)\| \leq \|\tilde{\beta}_3\|_\infty \sup_{s\in(-\infty,0]} \vartheta\left(\mathcal{U}_\eta(s)\right) + \varepsilon'; \quad \text{for a.e} \quad \eta \in \Lambda. \tag{32}$$

Now, we consider the mapping $\mathbb{N} : \mathfrak{J} \to \mathcal{C}$ defined by:

$$\begin{aligned}
\delta \in \Theta \longmapsto \mathbb{N}(\delta, \varrho) = e^{-\varpi(\psi(\eta)-\psi(0))}&[\chi(0) - \wp(0, \chi)] + \tilde{\rho}_\eta(\delta) \\
&+ \int_0^\eta \Psi^{\varpi;\psi}_\zeta(\eta, \tau)\psi'(\tau)\tilde{\sigma}_\tau(\delta)d\tau; \quad \eta \in \Lambda.
\end{aligned}$$

It is clear that $\mathbb{N}$ is continuous and $\mathbb{N}(\mathfrak{J}) \subset \mathbb{Z}(\mathcal{U})$. Additionally, by (21) and (22), given $y \in \mathcal{U}$, we can find

$\delta \in \mathfrak{J}$ such that:

$$\begin{aligned}
\|(\mathcal{P}y)(\eta) - \mathbb{N}(\delta,\eta)\| &\le \|\wp(\eta, x_\eta + y_\eta) - \tilde{\rho}_\eta(\delta)\| \\
&\quad + \int_0^\eta \Psi_\zeta^{\varpi;\psi}(\eta,\tau)\psi'(\tau)\|\aleph(\eta, x_\tau + y_\tau, \mathfrak{f}(\tau)) - \tilde{\sigma}_\tau(\delta)\| d\tau \\
&\le \|\tilde{\beta}_3\|_\infty \sup_{s\in(-\infty,0]} \vartheta\left(\mathcal{U}_\eta(s)\right) + \varepsilon' \\
&\quad + \int_0^\eta \frac{[\psi(\eta)-\psi(\tau)]^{\zeta-1}}{\Gamma(\zeta)}\psi'(\tau) \\
&\quad \times \left(\frac{\|\tilde{\beta}_1\|_{L^\infty}}{1-\|\tilde{\beta}_2\|_{L^\infty}} \sup_{s\in(-\infty,0]} \vartheta\left(\mathcal{U}_\eta(s)\right) + \epsilon \right) d\tau.
\end{aligned}$$

Using Lemma 2.10 and the features of ϑ, we obtain

$$\begin{aligned}
\|(\mathcal{P}y)(\eta) - \mathbb{N}(\delta,\eta)\| &\le \|\tilde{\beta}_3\|_{L^\infty}\vartheta(\mathcal{U}) + \varepsilon' \\
&\quad + \int_0^\eta \frac{[\psi(\eta)-\psi(\tau)]^{\zeta-1}}{\Gamma(\zeta)}\psi'(\tau) \left(\frac{\|\tilde{\beta}_1\|_{L^\infty}}{1-\|\tilde{\beta}_2\|_{L^\infty}}\vartheta(\mathcal{U}) + \epsilon \right) d\tau.
\end{aligned}$$

For every $\epsilon > 0$ and $\epsilon' > 0$, we conclude

$$\|(\mathcal{P}y)(\eta) - \mathbb{N}(\delta,\eta)\| \le \frac{\|\tilde{\beta}_3\|_{L^\infty} + \|\tilde{\beta}_1\|_{L^\infty}[\psi(\kappa)-\psi(0)]^\zeta}{(1-\|\tilde{\beta}_2\|_{L^\infty})\Gamma(\zeta+1)}\vartheta(\mathcal{U}).$$

Hence $\vartheta(\mathcal{P}\mathcal{U}) \le \lambda\vartheta(\mathcal{U})$, with $\lambda = \frac{\|\tilde{\beta}_3\|_{L^\infty} + \|\tilde{\beta}_1\|_{L^\infty}[\psi(\kappa)-\psi(0)]^\zeta}{(1-\|\tilde{\beta}_2\|_{L^\infty})\Gamma(\zeta+1)}$.

As a consequence of the three above steps, from Theorem 2.11, the operator $\mathcal{P}$ has at least fixed point y. Then, $\gamma = x + y$ is a fixed point to the operator $\mathbb{T}$, which is a solution of the problem (3)–(4). □

5. Implicit Neutral Tempered ψ-Caputo Fractional Differential Equations with State-Dependent Delay

5.1. The Finite Delay Case

Definition 5.1. *A solution of (5)-(6) is a function $\gamma \in \Xi$, defined by*

$$\gamma(\eta) = \begin{cases} e^{-\varpi(\psi(\eta)-\psi(0))}[\chi(0) - \wp(0,\chi)] + \wp(\eta, \gamma_{\varrho(\eta,\gamma_\eta)}) \\ + \int_0^\eta \Psi_\zeta^{\varpi;\psi}(\eta,\tau)\psi'(\tau)\aleph(\tau, \gamma_{\varrho(\tau,\gamma_\tau)}, \mathfrak{f}(\tau))d\tau; \quad \eta \in \Lambda, \\ \gamma(\eta) = \chi(\eta); \quad \eta \in [-\theta, 0], \end{cases} \tag{33}$$

where $\mathfrak{f} \in \mathfrak{C}$ satisfies the following functional equation

$$\mathfrak{f}(\eta) = \aleph(\eta, \gamma_{\varrho(\eta,\gamma_\eta)}, \mathfrak{f}(\eta)).$$

Set

$$\mathcal{Q}(\varrho^-) = \{\varrho(\eta,\gamma) : (\eta,\gamma) \in \Lambda \times \Pi_\theta,\ \varrho(\eta,\gamma) \le 0\}.$$

We assume that $\varrho : \Lambda \times \Pi_\theta \to \mathbb{R}$ is continuous. Additionally, we introduce the following hypothesis:

(H_χ) The function $\eta \to \chi_\eta$ is continuous from $\mathcal{Q}(\varrho^-)$ into Π_θ, and there exists a continuous and bounded function $\mathcal{L}^\chi : \mathcal{Q}(\varrho^-) \to (0,\infty)$ such that

$$\|\chi_\eta\|_{\Pi_\theta} \le \mathcal{L}^\chi(\eta)\|\chi\|_{\Pi_\theta}.$$

Remark 5.2. *The condition* (H_χ) *is frequently verified by functions continuous and bounded. For more details, see, for instance, [30].*

Lemma 5.3 ([28], Lemma 2.4)**.** *If* $\gamma : [-\theta, \kappa] \to \Xi$ *is a function such that* $\gamma_0 = \chi$*, then*

$$\|\gamma_\eta\|_{\Pi_\theta} \leq \mathcal{L}^\chi \|\chi\|_{\Pi_\theta} + \sup_{s \in [0, max\{0,\eta\}]} \|\gamma(s)\|, \quad \eta \in \mathcal{Q}(\varrho^-) \cup \Lambda,$$

where $\mathcal{L}^\chi = \sup_{\eta \in \mathcal{Q}(\varrho^-)} |\mathcal{L}^\chi(\eta)|.$

To prove our results on the existence, we introduce the following hypothesis:

(H_8) There exists a constant $\tilde{\mathcal{R}} > 0$ such that

$$\|\chi\|_{\Pi_\theta} + \|\xi\|_{L^\infty} [\varphi_3(\|\chi\|_{\Pi_\theta}) + \varphi_3(\mathcal{R})] + \frac{[\psi(\kappa) - \psi(0)]^\zeta \tilde{\varphi}(\tilde{\mathcal{R}})}{\Gamma(\zeta + 1)} \leq \tilde{\mathcal{R}}, \tag{34}$$

where $\tilde{\varphi}$ is the comparison function defined by

$$\tilde{\varphi}(\eta) = (Id - \|\nu\|_{L^\infty} \varphi_2)^{-1} (\|\mu\|_{L^\infty} \varphi_1(\mathcal{L}^\chi \|\chi\|_{\Pi_\theta} + \eta)).$$

Theorem 5.4. *Assume that the hypotheses* $(H_1) - (H_6)$*,* (H_χ)*,* (H_8) *and the condition*

$$0 < \frac{\|\beta_3\|_{L^\infty} + \|\beta_1\|_{L^\infty} [\psi(\kappa) - \psi(0)]^\zeta}{(1 - \|\beta_2\|_{L^\infty}) \Gamma(\zeta + 1)} < 1, \tag{35}$$

hold. Then the problem (5)-(6) has at least one solution defined on $[-\theta, \kappa]$.

Proof. Transform the problem (5)-(6) into a fixed point equation. Consider the operator $\tilde{\mathbb{Z}} : \mathfrak{C} \to \mathfrak{C}$ defined by:

$$(\tilde{\mathbb{Z}}\gamma)(\eta) = \begin{cases} e^{-\varpi(\psi(\eta) - \psi(0))} [\chi(0) - \wp(0, \chi)] + \wp(\eta, \gamma_{\varrho(\eta, \gamma_\eta)}) \\ \\ + \int_0^\eta \Psi_\zeta^{\varpi;\psi}(\eta, \tau) \psi'(\tau) \aleph(\tau, \gamma_{\varrho(\tau, \gamma_\tau)}, \mathfrak{f}(\tau)) d\tau; \quad \eta \in \Lambda, \\ \\ \gamma(\eta) = \chi(\eta); \quad \eta \in [-\theta, 0], \end{cases} \tag{36}$$

where $\mathfrak{f} \in \mathfrak{C}$ satisfies the following functional equation

$$\mathfrak{f}(\eta) = \aleph(\eta, \gamma_{\varrho(\eta, \gamma_\eta)}, \mathfrak{f}(\eta)).$$

Let

$$\tilde{\Theta} = \{\gamma \in \Pi, \ \|\gamma\|_\Pi \leq \tilde{\mathcal{R}}\}, \tag{37}$$

be the closed and convex ball in Π, where $\tilde{\mathcal{R}}$ satisfies the inequality (34).

We can prove as in Theorem 3.3 that the operators $\tilde{\mathbb{Z}}$ satisfies the conditions of Theorem 2.11. This implies that the operator $\tilde{\mathbb{Z}}$ has at least a fixed point which is a solution of problem (5)-(6). ☐

5.2. The Infinite Delay Case

Definition 5.5. *A solution of (7)-(8) is a function $\gamma \in \Xi$, defined by*

$$\gamma(\eta) = \begin{cases} e^{-\varpi(\psi(\eta)-\psi(0))}[\chi(0) - \wp(0,\chi)] + \wp(\eta, \gamma_{\varrho(\eta,\gamma_\eta)}) \\ + \int_0^\eta \Psi_\zeta^{\varpi;\psi}(\eta,\tau)\psi'(\tau)\aleph(\tau, \gamma_{\varrho(\tau,\gamma_\tau)}, \mathfrak{f}(\tau))d\tau; \quad \eta \in \Lambda, \\ \gamma(\eta) = \chi(\eta); \quad \eta \in (-\infty, 0], \end{cases} \tag{38}$$

where $\mathfrak{f} \in \mathfrak{C}$ satisfies the following functional equation

$$\mathfrak{f}(\eta) = \aleph(\tau, \gamma_{\varrho(\eta,\gamma_\eta)}, \mathfrak{f}(\eta)).$$

Set

$$\mathfrak{Q}(\varrho^-) = \{\varrho(\eta,\gamma) : (\eta,\chi) \in \Lambda \times \mathcal{F},\ \varrho(\eta,\gamma) \leq 0\}.$$

We assume that $\varrho : \Lambda \times \mathcal{F} \to \mathbb{R}$ is continuous. Additionally, we introduce the following hypothesis:

(h_χ) the function $\eta \to \chi_\eta$ is continuous from $\mathfrak{Q}(\varrho^-)$ into $\mathcal{F}$, and there exists a continuous and bounded function $\tilde{\mathcal{L}}^\chi : \mathfrak{Q}(\varrho^-) \to (0,\infty)$ such that

$$\|\chi_\eta\|_{\mathcal{F}} \leq \tilde{\mathcal{L}}^\chi(\eta)\|\chi\|_{\mathcal{F}}.$$

Remark 5.6. *The condition (h_χ) is frequently verified by functions continuous and bounded. For more details, see, for instance, [30].*

Lemma 5.7 ([28], Lemma 2.4)**.** *If $\gamma : (-\infty, \kappa] \to \Xi$ is a function such that $\gamma_0 = \chi$, then*

$$\|\gamma_\eta\|_{\mathcal{F}} \leq (M^* + \tilde{\mathcal{L}}^\chi)\|\chi\|_{\mathcal{F}} + L^* \sup_{\Lambda \in [0, max\{0,\eta\}]} \|\gamma(s)\|, \quad \eta \in \mathfrak{Q}(\varrho^-) \cup \Lambda,$$

where $\tilde{\mathcal{L}}^\chi = \sup\limits_{\eta \in \mathcal{R}(\varrho^-)} |\tilde{\mathcal{L}}^\chi(\eta)|$

To prove our results on the existence, we introduce the following hypothesis:

(h_7) There exists a constant $\tilde{\mathcal{R}}^* > 0$ such that

$$\|\tilde{\xi}\|_{L^\infty}\phi_3(\tilde{\Delta} + L^*\tilde{\mathcal{R}}^*) + \frac{[\psi(\kappa) - \psi(0)]^\zeta \tilde{\phi}(\tilde{\mathcal{R}}^*)}{\Gamma(\zeta+1)} \leq \tilde{\mathcal{R}}^*, \tag{39}$$

where $\tilde{\Delta} = \|\chi\|_{\mathcal{F}}(HL^* + M^* + \mathcal{L}^\chi) + L^*\|\tilde{\xi}\|_{L^\infty}\phi_3((H + M^* + \mathcal{L}^\chi)\|\chi\|_{\mathcal{F}})$, and ϕ is the comparison function defined by

$$\tilde{\phi}(\eta) = (Id - \|\tilde{\nu}\|_{L^\infty}\phi_2)^{-1}(\|\tilde{\mu}\|_{L^\infty}\phi_1(\tilde{\Delta} + L^*\eta)).$$

Theorem 5.8. *Assume that the hypotheses (h_1) − (h_6), (h_χ), (h_8) and the condition*

$$0 < \frac{\|\tilde{\beta_3}\|_{L^\infty} + \|\tilde{\beta_1}\|_{L^\infty}[\psi(\kappa) - \psi(0)]^\zeta}{(1 - \|\tilde{\beta_2}\|_{L^\infty})\Gamma(\zeta+1)} < 1, \tag{40}$$

hold. Then the problem (7)-(8) has at least one solution defined on $(-\infty, \kappa]$.

Proof. Transform the problem (7)-(8) into a fixed point equation. Consider the operator $\tilde{\mathbb{T}} : \mathfrak{C} \to \mathfrak{C}$ defined by:

$$(\tilde{\mathbb{T}}\gamma)(\eta) = \begin{cases} e^{-\varpi(\psi(\eta)-\psi(0))}[\chi(0) - \wp(0,\chi)] + \wp(\eta, \gamma_{\varrho(\eta,\gamma_\eta)}) \\ \\ + \int_0^\eta \Psi_\zeta^{\varpi;\psi}(\eta,\tau)\psi'(\tau)\aleph(\tau, \gamma_{\varrho(\tau,\gamma_\tau)}, \mathfrak{f}(\tau))d\tau; \quad \eta \in \Lambda, \\ \\ \gamma(\eta) = \chi(\eta); \quad \eta \in (-\infty, 0], \end{cases} \tag{41}$$

where $\mathfrak{f} \in \mathfrak{C}$ satisfies the following functional equation

$$\mathfrak{f}(\eta) = \aleph(\eta, \gamma_{\varrho(\eta,\gamma_\eta)}, \mathfrak{f}(\eta)).$$

If γ satisfies $\gamma(\eta) = (\tilde{\mathbb{T}}\gamma)(\eta)$, then there is similar transformation to that in the proof of Theorem 4.3, given the following decomposition $\gamma(\eta) = x^*(\eta) + y^*(\eta)$ for $\eta \in \Lambda$, which implies that $\gamma_\eta = x^*_\eta + y^*_\eta$ for every $\eta \in \Lambda$, and the function $y^*(\cdot)$ satisfies

$$\begin{aligned} y^*(\eta) = {} & \wp\left(\eta, x^*_{\varrho(\eta,x^*_\eta+y^*_\eta)} + y^*_{\varrho(\eta,x^*_\eta+y^*_\eta)}\right) \\ & + \int_0^\eta \Psi_\zeta^{\varpi;\psi}(\eta,\tau)\psi'(\tau)\aleph\left(\tau, x^*_{\varrho(\tau,x^*_\tau+y^*_\tau)} + y^*_{\varrho(\tau,x^*_\tau+y^*_\tau)}, \mathfrak{f}(\tau)\right) d\tau, \end{aligned}$$

where $\mathfrak{f} \in \mathfrak{C}$ satisfies the following functional equation

$$\mathfrak{f}(\eta) = \aleph(\eta, x^*_{\varrho(\eta,x^*_\eta+y^*_\eta)} + y^*_{\varrho(\eta,x^*_\eta+y^*_\eta)}, \mathfrak{f}(\eta)).$$

Defined the operator $\tilde{\mathcal{P}} : \Pi_0 \to \Pi_0$ by

$$\begin{aligned} \tilde{\mathcal{P}}y(\eta) = {} & \wp\left(\eta, x^*_{\varrho(\eta,x^*_\eta+y^*_\eta)} + y^*_{\varrho(\eta,x^*_\eta+y^*_\eta)}\right) \\ & + \int_0^\eta \Psi_\zeta^{\varpi;\psi}(\eta,\tau)\psi'(\tau)\aleph(\tau, x^*_{\varrho(\tau,x^*_\tau+y^*_\tau)} + y^*_{\varrho(\tau,x^*_\tau+y^*_\tau)}, \mathfrak{f}(\tau))d\tau. \end{aligned}$$

Obviously, the operator $\tilde{\mathbb{T}}$ has a fixed point is equivalent to $\tilde{\mathcal{P}}$ having a fixed point, and so we turn to proving that $\tilde{\mathcal{P}}$ has a fixed point. Let

$$\Theta^* = \{\gamma \in \Pi_0, \ \|\gamma\|_{\Pi_0} \leq \tilde{\mathcal{R}}^*\}, \tag{42}$$

where $\tilde{\mathcal{R}}^*$ satisfies the inequality (39).

We can prove as in Theorem 4.3 that the operator $\tilde{\mathcal{P}}$ satisfies the conditions of Theorem 2.11. This implies that the operator $\tilde{\mathcal{P}}$ has at least fixed point y^*, Then, $\gamma = x^* + y^*$ is a fixed point to the operator $\tilde{\mathbb{T}}$, which is a solution of the problem (7)-(8). □

6. Examples

Let

$$\Xi = c_0 = \left\{\gamma = (\gamma_1, \gamma_2, \cdots, \gamma_k, \cdots) : \gamma_k \to 0 \ (k \to \infty)\right\},$$

be the Banach space with the norm:

$$\|\gamma\| = \sup_{k \in \mathbb{N}^*} |\gamma_k|.$$

Let Σ be a Banach space with the norm $\|\cdot\|_\Sigma$, to be specified later.

Example 6.1. *Consider the following problem of implicit neutral fractional differential equation involving tempered ψ-Caputo fractional differential equations with delay:*

$$ {}^{C}_{0}\mathfrak{D}^{\frac{1}{2},2;\psi}_{\eta}\left(\gamma(\eta)-\wp(\eta,\gamma_\eta)\right)=\aleph\left(\eta,\gamma_\eta,\ {}^{C}_{0}\mathfrak{D}^{\frac{1}{2},2;\psi}_{\eta}\left(\gamma(\eta)-\wp(\eta,\gamma_\eta)\right)\right);\eta\in[0,1], \tag{43}$$

$$\gamma(\eta)=\eta+1;\qquad \eta\in\Upsilon, \tag{44}$$

where $\psi(\eta)=\eta^2$, *the function* $\aleph:\Lambda\times\Sigma\times c_0\to c_0$ *defined by:*

$$\aleph_k(\eta,\lambda_k,\gamma_k)=\frac{k^2}{(k+3)^2(\eta^2+5)}\left(2+\|\lambda_k\|_\Sigma+\|\gamma_k\|\right);\quad \eta\in[0,1],\ k\in\mathbb{N}^*,$$

and the function $\wp:\Lambda\times\Sigma\to c_0$ *defined by:*

$$\wp_k(\eta,\gamma_k)=\frac{k^2(4+\|\gamma_k\|_\Sigma)}{5(\eta+2)};\quad \eta\in[0,1],\ k\in\mathbb{N}^*.$$

Case 01. *We put* $\Upsilon=[-1,0]$ *and* $\Sigma=C([-1,0],\Xi)$.
Set

$$\lambda=(\lambda_1,\lambda_2,\cdots,\lambda_k,\cdots),\quad \gamma=(\gamma_1,\gamma_2,\cdots,\gamma_k,\cdots),$$
$$\aleph=(\aleph_1,\aleph_2,\cdots,\aleph_k,\cdots),\quad \wp=(\wp_1,\wp_2,\cdots,\wp_k,\cdots).$$

Clearly, the hypotheses $(H_1),(H_2)$ *are satisfied. In addition, for each* $\eta\in\Lambda$, $\lambda\in\Sigma$ *and* $\gamma\in c_0$, *we have:*

$$\|\aleph(\eta,\lambda,\gamma)\|\le\frac{1}{\eta^2+5}\left(2+\|\lambda\|_\Sigma+\|\gamma\|\right),$$

and

$$\|\wp(\eta,\lambda)\|\le\frac{1}{\eta+2}\left(\frac{4+\|\lambda\|_\Sigma}{5}\right).$$

Then the hypotheses (H_3) *and* (H_4) *are satisfied with*

$$\mu(\eta)=\nu(\eta)=\frac{1}{\eta^2+5},\ \xi(\eta)=\frac{1}{\eta+2};\qquad \eta\in\Lambda,$$

and

$$\varphi_1(h)=\varphi_2(h)=1+h,\ \varphi_3(h)=\frac{4+h}{5};\qquad h\in[0,\infty).$$

On the other hand, for any non-empty, bounded, and convex subset $\mathcal{V}\in c_0$ *and* $\eta\in\mathfrak{J}$ *fixed, , let* $\sigma_\eta:\mathfrak{J}\to c_0$ *be an* α_η*-dense curve in* $\mathcal{V}(\eta)$ *for some* $\alpha_\eta>0$. *Then,* $\rho_\eta(\Theta)\subset\mathcal{V}(\eta)$ *and for* $\gamma\in\mathcal{V}$ *there is* $\delta\in\mathfrak{J}$ *satisfying:* $\|\gamma(\eta)-\sigma_\eta(\delta)\|\le\alpha_\eta$, *and, for any non-empty, bounded, and convex subset* $\mathcal{U}\in\Sigma$ *and* $\eta\in\Lambda$ *fixed, let* $\rho_\eta:\mathfrak{J}\to c_0$ *be an* $\tilde{\alpha}_\eta$*-dense curve in* $\mathcal{U}(\eta)$ *for some* $\tilde{\alpha}_\eta>0$. *Then,* $\rho_\eta(\mathfrak{J})\subset\mathcal{U}(\eta)$ *and for* $\lambda\in\mathcal{U}$ *there is* $\delta\in\mathfrak{J}$ *satisfying:* $\|\lambda(\eta)-\rho_\eta(\delta)\|\le\tilde{\alpha}_\eta$.
Therefore, we have

$$\|\aleph(\eta,\lambda_\eta,\gamma(\eta))-\aleph(\eta,\rho_\eta(\delta),\sigma_\eta(\delta))\|\le\frac{1}{\eta^2+5}\left(\|\lambda_\eta-\rho_\eta(\delta)\|_\Sigma+\|\gamma(\eta)-\sigma_\eta(\delta)\|\right),$$

and

$$\|\wp(\eta,\lambda_\eta)-\wp(\eta,\rho_\eta(\iota))\|\le\frac{1}{5(\eta+2)}\|\lambda_\eta-\rho_\eta(\iota)\|_\Sigma.$$

Thus

$$\vartheta(\aleph(\eta,\mathcal{U},\mathcal{V}))\le\frac{1}{\eta^2+5}\left(\sup_{s\in[-1,0]}\vartheta\left(\mathcal{U}(s)\right)+\vartheta(\mathcal{V})\right),$$

and

$$\vartheta(\wp(\eta,\mathcal{U})) \le \frac{1}{5(\eta+2)} \sup_{s\in[-1,0]} \vartheta\left(\mathcal{U}(s)\right).$$

The hypotheses (H_5) *and* (H_6) *are satisfied with*

$$\beta_1(\eta) = \beta_2(\eta) = \frac{1}{\eta^2+5}, \quad \beta_3(\eta) = \frac{1}{5(\eta+2)}; \quad \eta \in \Lambda.$$

Condition (17) is satisfied

$$\frac{\|\beta_3\|_{L^\infty} + \|\beta_1\|_{L^\infty}[\psi(\eta)-\psi(0)]^\zeta}{(1-\|\beta_2\|_{L^\infty})\Gamma(\zeta+1)} \le \frac{3}{4\sqrt{\pi}} < 1.$$

The hypothesis (H_7) *is satisfied with*

$$\mathcal{R} \ge \frac{28\pi + 10\sqrt{\pi}}{8\sqrt{\pi}-5}.$$

Then by Theorem 2.11, the problem (1)-(2) has at least one solution defined on $[-1,1]$.

Case 02. *We put* $\Upsilon = (-\infty, 0]$ *and* $\Sigma = \mathcal{F}$.

The phase space $\mathcal{F}$ *be* C_ς, *the space for any real positive constant* ς, *defined by*

$$C_\varsigma = \{\chi \in C((-\infty,0],\Xi) : \lim_{s\to-\infty} e^{\varsigma s}\chi(s) \quad \text{exist in } \Xi\},$$

endowed with the norm

$$\|\chi\|_{C_\varsigma} = \sup_{s\in(-\infty,0]} \|\chi(s)\|.$$

Then in the space C_ς *axioms* (A_1)–(A_3) *are satisfied, with* $H=1, \quad L(t)=M(t)=1$, *(see [4]).* *Simple computations show that all conditions of Theorem 4.3 are satisfied. Hence, our problem (43)-(44) has at least a solution defined on* $(-\infty,1]$.

Example 6.2. *Consider the following problem of implicit neutral fractional differential equation involving tempered* ψ*-Caputo fractional differential equations with stat-dependent delay:*

$$\begin{gathered} {}_0^C\mathfrak{D}_\eta^{\frac{1}{2},2;\psi}\left(\gamma(\eta) - \wp(\eta,\gamma_{\varrho(\eta,\gamma_\eta)})\right) = \aleph\left(\eta, \gamma_{\varrho(\eta,\gamma_\eta)}, {}_0^C\mathfrak{D}_\eta^{\frac{1}{2},2;\psi}\left(\gamma(\eta) - \wp(\eta,\gamma_{\varrho(\eta,\gamma_\eta)})\right)\right); \\ \eta \in [0,1], \end{gathered} \tag{45}$$

$$\gamma(\eta) = \eta + 1; \qquad \eta \in \Upsilon, \tag{46}$$

where $\psi(\eta) = \eta^2$, *the function* $\aleph : \Lambda \times \Sigma \times c_0 \to c_0$ *defined by:*

$$\begin{gathered} \aleph_k(\eta,\lambda_k,\gamma_k(\eta)) = \frac{k^2}{(k+3)^2(\eta^2+5)}\left(2 + \|\lambda_k(\eta+\omega(\chi(\eta)))\|_\Sigma + \|\gamma_k\|\right); \\ \eta \in [0,1], \ n \in \mathbb{N}^*, \end{gathered}$$

and the function $\wp : \Lambda \times \Sigma \to c_0$ *defined by:*

$$\wp_k(\eta,\gamma_k) = \frac{k^2(4 + \|\gamma_k(\eta+\omega(\chi(\eta)))\|_\Sigma)}{5(\eta+2)}; \quad \eta \in [0,1], \ k \in \mathbb{N}^*,$$

where $\omega \in \Omega$ *defined by:*

$$\varrho(\eta,\chi) = \eta + \omega(\chi(\eta)); \quad (\eta,\chi) \in \Lambda \times \Sigma.$$

Case 01. *We put* $\Upsilon = [-1, 0]$, $\Omega = C(\Xi, [0, 1])$ *and* $\Sigma = C([-1, 0], \Xi)$.
Simple computations show that all conditions of Theorem 5.4 are satisfied. Hence, our problem (45)-(46) has at least a solution defined on $[-1, 1]$.

Case 02. *We put* $\Upsilon = (-\infty, 0]$, $\Omega = C(\Xi, [0, \infty))$ *and* $\Sigma = \mathcal{F}$.
$\mathcal{F} = C_\gamma$ *is the phase space defined in Example 6.1.*
Simple computations show that all conditions of Theorem 5.8 are satisfied. Hence, our problem (45)-(46) has at least a solution defined on $(-\infty, 1]$.

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