Advances in the Theory of Nonlinear Analysis and its Applications 7 (2023) No. 5, 107-129.
https://doi.org/10.17762/atnaa.v7.i5.329
Available online at https://atnaea.org/
Research Article

Advances in the Theory of Nonlinear Analysis and its Applications

ISSN: 2587-2648

Peer-Reviewed Scientific Journal

New complex (fuzzy) generalized metric spaces and an application to integral equations

Iqra Shereen[a], Quanita Kiran[b], Hassen Aydi[c]

[a]*Department of Mathematics, Quaid-i-Azam University, Islamabad, Pakistan*
[b]*School of Electrical Engineering and Computer Sciences (SEECS), National University of Sciences and Technology (NUST), Sector H-12, Islamabad, Pakistan*
[c]*Institut Supérieur dInformatique et des Techniques de Communication, Université de Sousse, H. Sousse 4000, Tunisia*
Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, Ga-Rankuwa, South Africa

Abstract

The concepts of a complex-valued new extended b-metric space, a complex-valued new extended rectangular b-metric space and a fuzzy rectangular b-metric space are initiated. We present some fixed point results in these settings via different contraction type mappings. We also give some examples and as an application we solve an integral equation.

Keywords: Fixed point, New complex (fuzzy) generalized metric space, integral equation
2010 MSC: 47H10; 54H25; 46S40

1. Introduction

Fixed point theory plays a vital role in mathematics and applied sciences, such as an optimization, mathematical models and economic theories. Also, this theory has been applied to show the existence and uniqueness of a solution of differential equations, integral equations and many other branches of mathematics, see [1, 2, 3]. In 1922, Banach gave the most fundamental and remarkable theorem known as the Banach Contraction Principal (B.C.P), which is an important tool to ensure both the existence and uniqueness of fixed points for contraction mappings from a complete metric space to itself. This theorem provides an illustration of the unifying power of functional analytic methods and the usefulness

Email addresses: `iqrashereen@math.qau.edu.pk` (Iqra Shereen), `quanita.kiran@seecs.edu.pk` (Quanita Kiran), `hassen.aydi@isima.rnu.tn` (Hassen Aydi)

Received September 22, 2023; Accepted: November 29, 2023; Online: December 27, 2023.
.

of fixed point theory in analysis. The important feature of the Banach contraction principle is that it gives the existence, uniqueness and sequence of the successive approximation converges to a solution of the

In 1965, Zadeh [4] initiated the development of the modified set theory known as fuzzy set theory, which is a tool that makes a possible the description of a vague manipulation within them. The basic idea of the fuzzy set is simple and natural.

In 1975, Kramosil and Michalak [5] introduced the fuzzy metric space. In 1994, the concept of a fuzzy metric space was modified by George and Veeramani [6] such that the Hausdorff topology is induced in modified fuzzy metric spaces. The study of fixed point theorems in fuzzy mathematics was investigated by Heilpern [7]. He introduced the concept of a fuzzy contraction mappings and established fixed point results for fuzzy contraction mappings in the framework of complete metric spaces. In 1988, Grabiec [8] generalized the Banach and Edilston fixed point theorems in fuzzy metric spaces. Bakhtin [9] and Czerwik [10, 11] introduced the b-metric spaces and proved some fixed point theorems for single-valued as well as multivalued mappings in b-metric spaces. Respectively, the notion of a b-metric space was re-introduced by Khamsi [12] and Hussain [13] with the name of a metric type space. On the other hand, a very important paper on fixed point theory in applied sciences is investigated by Ran and Reurings [14]. They established the results by using partial order sets. In 2016, Nadaban [15] introduced the fuzzy b-metric space which is a more general space.

In 2015, George al. introduced the mixed concept of a b -metric space and a rectangular metric space, named as a rectangular b -metric space [RbMS] which is not necessarily Hausdorff. This space generalized many previous spaces. In 2017, Kamran et al. [1] introduced the concept of an extended b-metric space and many results then have been proved in this space. The idea of extended rectangular b-metric spaces is presented in [22] and [23]. In 2011, Azam et al. [16] defined the notion of a complex-valued metric space which is more general than the well-known metric space. Shukla et al. [17] extended the concept of a fuzzy metric space to complex-valued fuzzy metric space and obtained some fixed point results in this space. Demir [18] introduced the concept of a complex-valued fuzzy b-metric space, generalizing both the notion of a complex-valued fuzzy metric space and the notion of a b-metric space. In 2014, Mukheimer [19] introduced the notion of complex valued b-metric space. Recently, Naimatullah et al. [21] defined the notion of complex-valued extended b-metric space. Here, in the second section, we extend this space by defining a complex-valued new extended b-metric space and prove common fixed point results using almost contractive mappings. In the third section, we give the concept of a complex-valued new extended rectangular b-metric space and present a related fixed point result. Also, we introduce a fuzzy version in a rectangular space, known as a complex-valued new extended fuzzy rectangular b-metric space as a generalization of complex-valued fuzzy extended rectangular b-metric spaces and prove some fixed point results with some applications.

2. A complex-valued new extended b-metric space

We recall some basic concepts that are necessary in the sequel. Most of these preliminaries are recorded from [16, 20, 21].

Definition 2.1. *Let $\mathbb{C}$ be the set of all complex numbers and $x_1, x_2 \in \mathbb{C}$. The partial order on $\mathbb{C}$ is defined as: $x_1 \preceq x_2$ if and only if $\mathrm{Re}(x_1) \leq \mathrm{Re}(x_2)$ and $\mathrm{Im}(x_1) \leq \mathrm{Im}(x_2)$. This implies that $x_1 \preceq x_2$ if one of the following conditions are fulfilled:*
(i) $\mathrm{Re}(x_1) = \mathrm{Re}(x_2)$, $\mathrm{Im}(x_1) < \mathrm{Im}(x_2)$;
(ii) $\mathrm{Re}(x_1) < \mathrm{Re}(x_2)$, $\mathrm{Im}(x_1) = \mathrm{Im}(x_2)$;
(iii)$\mathrm{Re}(x_1) < \mathrm{Re}(x_2)$, $\mathrm{Im}(x_1) < \mathrm{Im}(x_2)$;
(iv)$\mathrm{Re}(x_1) = \mathrm{Re}(x_2)$, $\mathrm{Im}(x_1) = \mathrm{Im}(x_2)$.

In 2011, Azam et al. [16] introduced a complex-valued metric space as follows:

Definition 2.2. *Let X be a non-empty set. The function $d : X \times X \to \mathbb{C}$ is called a complex valued metric if for all $x, y, z \in X$, the following properties hold:*

(i) $0 \preceq d(x, y)$ *and* $d(x, y) = 0$ *iff* $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x,z) \preceq d(x, y) + d(y,z)$.
The pair (X,d) *is called a complex-valued metric space.*

Example 2.1. *Let* $X = [0,1]$. *For* $x, y \in X$, $d : X \times X \to \mathbb{C}$ *is defined as*

$$d(x, y) = \begin{cases} 0 & \text{if} \quad x = y; \\ i/2 & \text{if} \quad x \neq y. \end{cases}$$

Then d *is a complex-valued metric on* X.

In 2014, Mukheimer [19] introduced the notion of a complex-valued b-metric space as follows.

Definition 2.3. *Let* X *be a non-empty set and* $b \geq 1$ *be a real number. The function* $d : X \times X \to \mathbb{C}$ *is called a complex-valued b-metric if for all* $x, y, z \in X$, *the following properties hold:*
(i) $0 \preceq d(x, y)$ *and* $d(x, y) = 0$ *iff* $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x,z) \preceq b[d(x,y) + d(y,z)]$.
The pair (X, d) *is called a complex-valued b-metric space.*

Recently, Naimatullah et al. [21] defined the notion of a complex-valued extended b-metric space in the following way.

Definition 2.4. *Let* X *be a non-empty set and* $\theta : X \times X \to [1, \infty)$ *be a given function. The function* $d_\theta : X \times X \to \mathbb{C}$ *is called a complex valued extended b-metric if for all* $x, y, z \in X$, *the following properties hold:*
(i) $0 \preceq d_\theta(x,y)$ *and* $d_\theta(x, y) = 0$ *iff* $x = y$;
(ii) $d_\theta(x, y) = d_\theta(y, x)$;
(iii) $d_\theta(x, z) \preceq \theta(x,y)[d_\theta(x, y) + d_\theta(y, z)]$.
The pair (X, d_θ) *is called a complex-valued extended b-metric space.*

Example 2.2. *Let* $X = [0, \infty)$ *and* $\theta : X \times X \times X \to [0, \infty)$ *be defined as* $\theta(x, y, z) = \frac{1+ x+y}{x+ y}$.
Also,
(i)$d_\theta(x, y) = \frac{i}{xy}, \forall x, y \in (0,1]$,
(ii)$d_\theta(x, y) = 0$ *iff* $x = y \ \forall \ x, y \in [0, 1]$,
(iii)$d_\theta(x,0) = d_\theta(0,x) \forall \ x \in (0,1]$.
Then (X, d_θ) *is a complex-valued extended b-metric space.*

Definition 2.5. *[21] Let* (X,d) *be a complex-valued extended b-metric space. We denote*

$$s(u) = \{z \in \mathbb{C} : u \preceq z \}.$$

For $x \in X$ *and* $Y \in CB(X)$,

$$s(x, Y) = \bigcup_{y \in Y} s(d(x,y)) = \bigcup_{y \in Y} \{z \in \mathbb{C} : d(x, y) \preceq z\}.$$

For $Y, Y' \in CB(X)$,

$$s(Y', Y) = \left(\bigcap_{y \in Y} s(d(y, Y')) \right) \cup \left(\bigcap_{y' \in Y'} s(d(y', Y)) \right).$$

Now, we define a complex-valued new extended b-metric space which generalizes a complex-valued extended b-metric space. We give an example and a common fixed point result in this space using almost contraction mappings.

Definition 2.6. *Let X be a non-empty set and $\theta : X \times X \times X \to [0, \infty)$ be a function. Then $d_\theta : X \times X \to \mathbb{C}$ is called a complex-valued new extended b-metric if for all $x, y, z \in X$, the following properties hold:*
(i) $0 \preceq d_\theta(x, y)$ and $d_\theta(x, y) = 0$ iff $x = y$;
(ii) $d_\theta(x, y) = d_\theta(y, x)$;
(iii) $d_\theta(x, z) \preceq \theta(x, y, z)[d_\theta(x, y) + d_\theta(y, z)]$.
The pair (X, d_θ) is called a complex-valued new extended b-metric space.

Example 2.3. *Let $X = [0, \infty)$ and $\theta : X \times X \times X \to [0, \infty)$ be a function defined as $\theta(x, y, z) = 1 + x + y + z$. Also, for $x, y \in X$, $d_\theta : X \times X \to \mathbb{C}$ is defined as*

$$d_\theta(x, y) = \begin{cases} 0 & \text{if} \quad x = y; \\ i & \text{if} \quad x \neq y. \end{cases}$$

Then d_θ is a complex-valued new extended b-metric on X.

We prove our main result using the following definition.

Definition 2.7. *Let (X, d_θ) be a complex-valued new extended b-metric space. A mapping $T : X \to X$ is said to be an almost contraction if there exist $r \in (0, 1)$ and $t \geq 0$ such that for all $x, y \in X$*

$$d_\theta(Tx, Ty) \leq r d_\theta(x, y) + t d_\theta(y, Tx).$$

Theorem 2.1. *Let (X, d_θ) be a complex-valued new extended b-metric space and X be a non-empty set , $\theta : X \times X \times X \to [0, \infty)$ be a function and $S, T : X \times X \to CB(X)$ be a pair of multi-valued mappings with g.l.b property such that for all $x, y \in X$*

$$a_1 d_\theta(x, y) + a_2[d_\theta(x, Sx) + d_\theta(y, Ty)] + a_3[d_\theta(y, Sx) + d_\theta(x, Ty)]$$

$$+a_4 \frac{d_\theta(y, Ty)\ [1 + d_\theta(x, Sx)]}{1 + d_\theta(x, y)} + a_5 \frac{d_\theta(y, Sx)[1 + d_\theta(x, Ty)]}{1 + d_\theta(x, y)}$$

$$+a_6 \frac{d_\theta(x, y)[1 + d_\theta(x, Sx) + d_\theta(y, Sx)]}{1 + d_\theta(x, y)} + t d_\theta(y, Sx) \in s(Sx, Ty) \tag{1}$$

where $a_1, a_2, a_3, a_4, a_5, a_6, t$ are non-negative real numbers such that $a_1 + 2a_2 + 2a_3\theta(x_0, x_1, x_2) + a_4 + a_6 < 1$ and $k(1 - a_2 - a_3\theta(x_0, x_1, x_2) - a_4 = a_1 + a_2 + a_3\theta(x_0, x_1, x_2) + a_6$ where $k \in [0, \infty)$ be such that for each $x_0 \in X$, $\lim_{n,m\to\infty} \theta(x_n, x_{n+1}, x_m) < 1/k$. Also, $\lim_{n\to\infty} \theta(x, x_n, Tx)$ and $\lim_{n\to\infty} \theta(x, x_n, x_{n+1})$ exist and are finite, then S and T have a unique common fixed point.

Proof. Let $x_0 \in X$ then $Tx_0 \neq \emptyset$. Let $x_1 \in Sx_0$, then

$$a_1 d_\theta(x_0, x_1) + a_2[d_\theta(x_0, Sx_0) + d_\theta(x_1, Tx_1)] + a_3[d_\theta(x_1, Sx_0) + d_\theta(x_0, Tx_1)]$$

$$+a_4 \frac{d_\theta(x_1, Tx_1)[1 + d_\theta(x_0, Sx_0)]}{1 + d_\theta(x_0, x_1)} + a_5 \frac{d_\theta(x_1, Sx_0)[1 + d_\theta(x_0, Tx_1)]}{1 + d_\theta(x_0, x_1)}$$

$$+a_6 \frac{d_\theta(x_0, x_1)[1 + d_\theta(x_0, Sx_0) + d_\theta(x_1, Sx_0)]}{1 + d_\theta(x_0, x_1)} + t d_\theta(x_1, Sx_0) \in s(Sx_0, Tx_1)$$

$$a_1 d_\theta(x_0, x_1) + a_2[d_\theta(x_0, Sx_0) + d_\theta(x_1, Tx_1)] + a_3[d_\theta(x_1, Sx_0) + d_\theta(x_0, Tx_1)]$$

$$+a_4 \frac{d_\theta(x_1, Tx_1)[1 + d_\theta(x_0, Sx_0)]}{1 + d_\theta(x_0, x_1)} + a_5 \frac{d_\theta(x_1, Sx_0)[1 + d_\theta(x_0, Tx_1)]}{1 + d_\theta(x_0, x_1)}$$

$$+a_6 \frac{d_\theta(x_0, x_1)[1 + d_\theta(x_0, Sx_0) + d_\theta(x_1, Sx_0)]}{1 + d_\theta(x_0, x_1)} + t d_\theta(x_1, Sx_0) \in \bigcap_{a \in Sx_0} s(a, Tx_1).$$

This implies that

$$a_1 d_\theta(x_0, x_1) + a_2[d_\theta(x_0, Sx_0) + d_\theta(x_1, Tx_1)] + a_3[d_\theta(x_1, Sx_0) + d_\theta(x_0, Tx_1)]$$

$$+a_4 \frac{d_\theta(x_1, Tx_1)[1 + d_\theta(x_0, Sx_0)]}{1 + d_\theta(x_0, x_1)} + a_5 \frac{d_\theta(x_1, Sx_0)[1 + d_\theta(x_0, Tx_1)]}{1 + d_\theta(x_0, x_1)}$$

$$+a_6 \frac{d_\theta(x_0, x_1)[1 + d_\theta(x_0, Sx_0) + d_\theta(x_1, Sx_0)]}{1 + d_\theta(x_0, x_1)} + t d_\theta(x_1, Sx_0) \in s(a, Tx_1), \qquad \forall a \in Sx_0.$$

Since $x_1 \in Sx_0$, one writes

$$a_1 d_\theta(x_0, x_1) + a_2[d_\theta(x_0, Sx_0) + d_\theta(x_1, Tx_1)] + a_3[d_\theta(x_1, Sx_0) + d_\theta(x_0, Tx_1)]$$

$$+a_4 \frac{d_\theta(x_1, Tx_1)[1 + d_\theta(x_0, Sx_0)]}{1 + d_\theta(x_0, x_1)} + a_5 \frac{d_\theta(x_1, Sx_0)[1 + d_\theta(x_0, Tx_1)]}{1 + d_\theta(x_0, x_1)}$$

$$+a_6 \frac{d_\theta(x_0, x_1)[1 + d_\theta(x_0, Sx_0) + d_\theta(x_1, Sx_0)]}{1 + d_\theta(x_0, x_1)} + t d_\theta(x_1, Sx_0) \in s(Sx_1, Tx_1)$$

$$a_1 d_\theta(x_0, x_1) + a_2[d_\theta(x_0, Sx_0) + d_\theta(x_1, Tx_1)] + a_3[d_\theta(x_1, Sx_0) + d_\theta(x_0, Tx_1)]$$

$$+a_4 \frac{d_\theta(x_1, Tx_1)[1 + d_\theta(x_0, Sx_0)]}{1 + d_\theta(x_0, x_1)} + a_5 \frac{d_\theta(x_1, Sx_0)[1 + d_\theta(x_0, Tx_1)]}{1 + d_\theta(x_0, x_1)}$$

$$+a_6 \frac{d_\theta(x_0, x_1)[1 + d_\theta(x_0, Sx_0) + d_\theta(x_1, Sx_0)]}{1 + d_\theta(x_0, x_1)} + t d_\theta(x_1, Sx_0) \in \bigcup_{b \in Tx_1} s(x_1, b).$$

There exists $x_2 \in Tx_1$ such that

$$a_1 d_\theta(x_0, x_1) + a_2[d_\theta(x_0, Sx_0) + d_\theta(x_1, Tx_1)] + a_3[d_\theta(x_1, Sx_0) + d_\theta(x_0, Tx_1)]$$

$$+a_4 \frac{d_\theta(x_1, Tx_1)[1 + d_\theta(x_0, Sx_0)]}{1 + d_\theta(x_0, x_1)} + a_5 \frac{d_\theta(x_1, Sx_0)[1 + d_\theta(x_0, Tx_1)]}{1 + d_\theta(x_0, x_1)}$$

$$+a_6 \frac{d_\theta(x_0, x_1)[1 + d_\theta(x_0, Sx_0) + d_\theta(x_1, Sx_0)]}{1 + d_\theta(x_0, x_1)} + t d_\theta(x_1, Sx_0) \in s(d(x_1, x_2)).$$

By definition and g.l.b property of S and T, we have

$$d_\theta(x_1,x_2) \preceq a_1 d_\theta(x_0,x_1) + a_2[d_\theta(x_0,Sx_0) + d_\theta(x_1,Tx_1)] + a_3[d_\theta(x_1, Sx_0) + d_\theta(x_0, Tx_1)]$$

$$+a_4 \frac{d_\theta(x_1, Tx_1)[1 + d_\theta(x_0, Sx_0)]}{1 + d_\theta(x_0,x_1)} + a_5 \frac{d_\theta(x_1,Sx_0)[1 + d_\theta(x_0,Tx_1)]}{1 + d_\theta(x_0, x_1)}$$

$$+a_6 \frac{d_\theta(x_0, x_1)[1 + d_\theta(x_0,Sx_0) + d_\theta(x_1,Sx_0)]}{1 + d_\theta(x_0, x_1)} + t d_\theta(x_1,Sx_0)$$

$$d_\theta(x_1, x_2) \preceq a_1 d_\theta(x_0, x_1) + a_2[d_\theta(x_0, x_1) + d_\theta(x_1, x_2)] + a_3[d_\theta(x_1,x_1) + d_\theta(x_0,x_2)]$$

$$+a_4 \frac{d_\theta(x_1,x_2)[1 + d_\theta(x_0, x_1)]}{1 + d_\theta(x_0,x_1)} + a_5 \frac{d_\theta(x_1, x_1)[1 + d_\theta(x_0,x_2)]}{1 + d_\theta(x_0, x_1)}$$

$$+a_6 \frac{d_\theta(x_0, x_1)[1 + d_\theta(x_0, x_1) + d_\theta(x_1,x_1)]}{1 + d_\theta(x_0, x_1)} + t d_\theta(x_1, x_1).$$

That is,

$$|d_\theta(x_1,x_2)| \leq a_1|d_\theta(x_0,x_1)| + a_2[|d_\theta(x_0, x_1)| + |d_\theta(x_1, x_2)|]$$

$$+a_3|d_\theta(x_0, x_2)| + a_4|d_\theta(x_1, x_2)| + a_6|d_\theta(x_0, x_1)|.$$

Since $|d_\theta(x_1, x_2)| < 1 + |d_\theta(x_1, x_2)|$

$$|d_\theta(x_1, x_2)| \leq \frac{a_1 + a_2 + a_3\theta(x_0,x_1,x_2) + a_6}{1 - a_2 - a_3\theta(x_0,x_1, x_2) - a_4}|d_\theta(x_0,x_1)|$$

which further implies

$$\begin{aligned}
|d_\theta(x_1,x_2)| &\leq k|d_\theta(x_0,x_1)| \\
&. \\
&. \\
|d_\theta(x_n, x_{n+1})| &\leq k^n|d_\theta(x_0, x_1)|.
\end{aligned}$$

For $m > n$, we have

$$\begin{aligned}
d_\theta(x_n, x_m) &\preceq \theta(x_n, x_{n+1}, x_m)[d_\theta(x_n, x_{n+1}) + d_\theta(x_{n+1}, x_m)] \\
&\preceq \theta(x_n, x_{n+1}, x_m)[d_\theta(x_n, x_{n+1}) + \theta(x_{n+1}, x_{n+2},x_m)[d_\theta(x_{n+1}, x_{n+2}) + d_\theta(x_{n+2},x_m)] \\
&\preceq \theta(x_n, x_{n+1},x_m)d_\theta(x_n, x_{n+1}) + \theta(x_n, x_{n+1},x_m)\theta(x_{n+1}, x_{n+2},x_m)d_\theta(x_{n+1}, x_{n+2}) \\
&\quad +... + \theta(x_n, x_{n+1},x_m)\theta(x_{n+1}, x_{n+2},x_m)...\theta(x_{m-2}, x_{m-1},x_m)d_\theta(x_{m-1},x_m) \\
&\preceq \theta(x_n, x_{n+1}, x_m)k^n d_\theta(x_0, x_1) + \theta(x_n, x_{n+1}, x_m)\ \theta(x_{n+1}, x_{n+2}, x_m)k^{n+1}d_\theta(x_0,x_1) \\
&\quad +... + \theta(x_n, x_{n+1}, x_m)\theta(x_{n+1}, x_{n+2},x_m)\ ...\ \theta(x_{m-2}, x_{m-1}, x_m)k^{m-1}d_\theta(x_0, x_1) \\
&\preceq \theta(x_1, x_2, x_m)\theta(x_2, x_3, x_m)\ ...\ \theta(x_n, x_{n+1}, x_m)k^n d_\theta(x_0, x_1) \\
&\quad +\theta(x_1, x_2,x_m)\ \theta(x_2, x_3, x_m)\ ...\ \theta(x_n, x_{n+1}, x_m)\theta(x_{n+1}, x_{n+2}, x_m)k^{n+1}d_\theta(x_0, x_1) \\
&\quad +... + \theta(x_1, x_2, x_m)\ \theta(x_2, x_3, x_m)\ ...\ \theta(x_{m-2}, x_{m-1}, x_m)k^{m-1}d_\theta(x_0, x_1).
\end{aligned}$$

Since $\lim\limits_{n,m\to\infty} k\theta(x_n,\ x_{n+1},\ x_m) < 1$, the series $\sum_{n=1}^{\infty} k^n \Pi_{i=1}^{n}\theta(x_i, x_{i+1}, x_m)$ converges for each $m \in \mathbb{N}$ by ratio test.
Let $S = \sum_{n=1}^{\infty} k^n\ \Pi_{i=1}^{n}\theta(x_i, x_{i+1}, x_m)$ and $S_n = \sum_{j=1}^{n} k^j \Pi_{i=1}^{n}\theta(x_i, x_{i+1}, x_m)$. For $m > n$, we have

$$d_\theta(x_n,\ x_m) \preceq d_\theta(x_0,\ x_1)[S_{m-1} - S_n].$$

Equivalently,

$$|d_\theta(x_n,\ x_m)| \leq |d_\theta(x_0,\ x_1)|[S_{m-1} - S_n].$$

When $n, m \to \infty$,

$$|d_\theta(x_n, x_m)| \longrightarrow 0$$

which shows that $\{x_n\}$ is a Cauchy sequence in X. By completeness of X, there exists some $x \in X$ such that $\lim\limits_{n\to\infty} x_n = x$.
Now, we show that $x \in Sx$ and $x \in Tx$. Using 1, we have,

$$a_1 d_\theta(x_{2n}, x) + a_2[d_\theta(x_{2n}, Sx_{2n}) + d_\theta(x,\ Tx)] + a_3[d_\theta(\ x, Sx_{2n}) + d_\theta(x_{2n}, Tx)]$$

$$+\ a_4 \frac{d_\theta(\ x,\ Tx)[1 + d_\theta(x_{2n}, Sx_{2n})]}{1 + d_\theta(x_{2n}, x)} + a_5 \frac{d_\theta(\ x, Sx_{2n})[1 + d_\theta(x_n, Tx)]}{1 + d_\theta(\ x_{2n}, x)}$$

$$+a_6 \frac{d_\theta(x_{2n}, x)[1 + d_\theta(x_{2n}, Sx_{2n}) + d_\theta(\ x, Sx_{2n})]}{1 + d_\theta(x_{2n}, x)} + t d_\theta(x, Sx_{2n}) \in s(Sx_{2n}, Tx)$$

which implies

$$a_1 d_\theta(x_{2n}, x) + a_2[d_\theta(x_{2n}, Sx_{2n}) + d_\theta(x,\ Tx)] + a_3[d_\theta(x, Sx_{2n}) + d_\theta(x_{2n}, Tx)]$$

$$+\ a_4 \frac{d_\theta(x,\ Tx)[1 + d_\theta(x_{2n}, Sx_{2n})]}{1 + d_\theta(x_{2n}, x)} + a_5 \frac{d_\theta(x, Sx_{2n})[1 + d_\theta(x_n, Tx)]}{1 + d_\theta(x_{2n}, x)}$$

$$+a_6 \frac{d_\theta(x_{2n}, x)[1 + d_\theta(x_{2n}, Sx_{2n}) + d_\theta(x, Sx_{2n})]}{1 + d_\theta(x_{2n}, x)} + t d_\theta(x, Sx_{2n}) \in \bigcap_{a\in Sx_{2n}} s(a, Tx)$$

$$a_1 d_\theta(x_{2n}, x) + a_2[d_\theta(x_{2n}, Sx_{2n}) + d_\theta(x,\ Tx)] + a_3[d_\theta(x, Sx_{2n}) + d_\theta(x_{2n}, Tx)]$$

$$+\ a_4 \frac{d_\theta(x,\ Tx)[1 + d_\theta(x_{2n}, Sx_{2n})]}{1 + d_\theta(x_{2n}, x)} + a_5 \frac{d_\theta(x, Sx_{2n})[1 + d_\theta(x_n, Tx)]}{1 + d_\theta(x_{2n}, x)}$$

$$+a_6 \frac{d_\theta(x_{2n}, x)[1 + d_\theta(x_{2n}, Sx_{2n}) + d_\theta(x, Sx_{2n})]}{1 + d_\theta(x_{2n}, x)} + t d_\theta(x, Sx_{2n}) \in s(a, Tx), \qquad \forall a \in Sx_{2n}.$$

Also $x_{2n+1} \in Sx_{2n}$, therefore we have

$$a_1 d_\theta(x_{2n}, x) + a_2[d_\theta(x_{2n}, Sx_{2n}) + d_\theta(x,\ Tx)] + a_3[d_\theta(\ x, Sx_{2n}) + d_\theta(x_{2n}, Tx)]$$

$$+\ a_4 \frac{d_\theta(\ x,\ Tx)[1 + d_\theta(x_{2n}, Sx_{2n})]}{1 + d_\theta(\ x_{2n}, x)} + a_5 \frac{d_\theta(x, Sx_{2n})[1 + d_\theta(x_n, Tx)]}{1 + d_\theta(x_{2n}, x)}$$

$$+a_6\frac{d_\theta(x_{2n},x)[1+ d_\theta(x_{2n},Sx_{2n})+d_\theta(x,Sx_{2n})]}{1+ d_\theta(x_{2n},x)}+td_\theta(x,Sx_{2n})\in s(x_{2n+1},Tx)$$

$$a_1d_\theta(x_{2n},x)+a_2[d_\theta(x_{2n},Sx_{2n})+d_\theta(x, Tx)]+ a_3[d_\theta(x,Sx_{2n})+d_\theta(x_{2n},Tx)]$$

$$+ a_4\frac{d_\theta(x, Tx)[1+d_\theta(x_{2n},Sx_{2n})]}{1+d_\theta(x_{2n},x)}+a_5\frac{d_\theta(x,Sx_{2n})[1+ d_\theta(x_n,Tx)]}{1+d_\theta(x_{2n},x)}$$

$$+a_6\frac{d_\theta(x_{2n},x)[1+d_\theta(x_{2n},Sx_{2n})+d_\theta(x,Sx_{2n})]}{1+d_\theta(x_{2n},x)}+td_\theta(x,Sx_{2n})\in \bigcup_{b\in Tx}s(d_\theta(x_{2n+1},b).$$

It implies that there exists some $x_n\in Tx$ such that

$$a_1d_\theta(x_{2n},x)+a_2[d_\theta(x_{2n},Sx_{2n})+d_\theta(x, Tx)]+ a_3[d_\theta(x,Sx_{2n})+d_\theta(x_{2n},Tx)]$$

$$+ a_4\frac{d_\theta(x, Tx)[1+d_\theta(x_{2n},Sx_{2n})]}{1+d_\theta(x_{2n},x)}+a_5\frac{d_\theta(x,Sx_{2n})[1+ d_\theta(x_n,Tx)]}{1+d_\theta(x_{2n},x)}$$

$$+a_6\frac{d_\theta(x_{2n},x)[1+d_\theta(x_{2n},Sx_{2n})+d_\theta(x,Sx_{2n})]}{1+d_\theta(x_{2n},x)}+td_\theta(x,Sx_{2n})\in s(d_\theta(x_{2n+1},x_n)$$

$$d_\theta(x_{2n+1},x_n)\preceq a_1d_\theta(x_{2n},x)+a_2[d_\theta(x_{2n},Sx_{2n})+d_\theta(x, Tx)]+a_3[d_\theta(x,Sx_{2n})+d_\theta(x_{2n},Tx)]$$

$$+a_4\frac{d_\theta(x,Tx)[1+ d_\theta(x_{2n},Sx_{2n})]}{1+ d_\theta(x_{2n},x)}+a_5\frac{d_\theta(x,Sx_{2n})[1+ d_\theta(x_n,Tx)]}{1+d_\theta(x_{2n},x)}$$

$$+a_6\frac{d_\theta(x_{2n},x)[1+d_\theta(x_{2n},Sx_{2n})+d_\theta(x,Sx_{2n})]}{1+ d_\theta(x_{2n},x)}+td_\theta(x,Sx_{2n})$$

which further implies

$$d_\theta(x_{2n+1},x_n)\preceq a_1d_\theta(x_{2n},x)+a_2[d_\theta(x_{2n}, x_{2n+1})+d_\theta(x,x_n)]+a_3[d_\theta(x,x_{2n+1})+d_\theta(x_{2n},x_n)]$$

$$+a_4\frac{d_\theta(x,x_n)[1+d_\theta(x_{2n}, x_{2n+1})]}{1+d_\theta(x_{2n},x)}+a_5\frac{d_\theta(x,x_{2n+1})[1+d_\theta(x_n,x_n)]}{1+ d_\theta(x_{2n},x)}$$

$$+a_6\frac{d_\theta(x_{2n},x)[1+d_\theta(x_{2n}, x_{2n+1})+ d_\theta(x,x_{2n+1})]}{1+d_\theta(x_{2n},x)}+td_\theta(x,x_{2n+1}).$$

Now, we have

$$\begin{aligned}
&d_\theta (x,x_n)\preceq\theta (x,x_{2n+1}, x_n)[d_\theta (x,x_{2n+1})+ d_\theta(x_{2n+1},x_n)]\\
\preceq\ &\theta (x_n,x_{2n+1},x_n) [d_\theta(x, x_{2n+1})+\theta(x, x_{2n+1},x_n)a_1d_\theta(x_{2n},x)\\
&+\theta(x,x_{2n+1},x_n)a_2[d_\theta(x_{2n},x_{2n+1})+d_\theta(x,x_n)]+\theta(x, x_{2n+1}, x_n)a_3[d_\theta(x, x_{2n+1})+d_\theta(x_{2n},x_n)]\\
&+\theta(x,x_{2n+1},x_n)a_4\frac{d_\theta(x,x_n)[1+d_\theta(x_{2n},x_{2n+1})]}{1+d_\theta(x_{2n},x)}+\theta(x,x_{2n+1}, x_n)a_5\frac{d_\theta(x,x_{2n+1})[1+d_\theta(x_n, x_n)]}{1+ d_\theta(x_{2n},x)}\\
&+\theta(x,x_{2n+1}, x_n)a_6\frac{d_\theta(x_{2n},x)[1+d_\theta(x_{2n}, x_{2n+1})+d_\theta(x, x_{2n+1})]}{1+d_\theta(x_{2n},x)}+\theta(x,x_{2n+1}, x_n)td_\theta(x, x_{2n+1})
\end{aligned}$$

which implies

$$\begin{aligned} d_\theta(x,x_n) &\leq |d_\theta(x, x_{2n+1})| + a_1|d_\theta(x_{2n},x)| + a_2[|d_\theta(x_{2n}, x_{2n+1}) + d_\theta(x,x_n)|] \\ &+a_3[|d_\theta(x,x_{2n+1})| + |d_\theta(x_{2n},x_n)|] + a_4\left|\frac{d_\theta(x,x_n)[1+ d_\theta(x_{2n},x_{2n+1})]}{1+ d_\theta(x_{2n},x)}\right| \\ &+a_5\left|\frac{d_\theta(x,x_{2n+1})[1+ d_\theta(x_n,x_n)]}{1+ d_\theta(x_{2n},x)}\right| \\ &+a_6\left|\frac{d_\theta(x_{2n},x)[1+d_\theta(x_{2n},x_{2n+1}) + d_\theta(x, x_{2n+1})]}{1+d_\theta(x_{2n},x)}\right| + t|d_\theta(x, x_{2n+1})|. \end{aligned}$$

By letting $n \to \infty$, we get

$$d_\theta(x, x_n) \to 0 \text{ when } n \to \infty$$

that is, $x_n \to x$ as $n \to \infty$. Since Tx is closed , we have $x \in Tx$. Similarly, $x \in Sx$. Hence, T and S have a common fixed point. □

If $S = T$, we have the following corollary.

Corollary 2.1. *Let (X, d_θ) be a complex-valued new extended b-metric space and X be a non-empty set, $\theta : X \times X \times X \to [0,\infty)$ be a function and $T : X \times X \to CB(X)$ be a multi-valued mapping with g.l.b property such that for all $x, y \in X$*

$$a_1 d_\theta(x, y) + a_2[d_\theta(x, Tx) + d_\theta(y, Ty)] + a_3[d_\theta(y, Tx) + d_\theta(x, Ty)]$$

$$+a_4\frac{d_\theta(y, Ty)[1+ d_\theta(x, Tx)]}{1+ d_\theta(x, y)} + a_5\frac{d_\theta(y, Tx)[1+ d_\theta(x, Ty)]}{1+ d_\theta(x, y)}$$

$$+a_6\frac{d_\theta(x, y)[1+ d_\theta(x, Tx) + d_\theta(y, Tx)]}{1+ d_\theta(x,y)} + td_\theta(y,Tx) \in s(Tx, Ty) \tag{2}$$

where $a_1, a_2, a_3, a_4, a_5, a_6, t$ are non-negative real numbers such that $a_1 + 2a_2 + 2a_3\theta(x_0,x_1,x_2) + a_4 + a_6 < 1$ and $k(1 - a_2 - a_3\theta(x_0,x_1,x_2) - a_4 = a_1 + a_2 + a_3\theta(x_0,x_1,x_2) + a_6$ where $k \in [0,\infty)$ is such that for each $x_0 \in X$, $\lim_{n,m\to\infty}\theta(x_n, x_{n+1}, x_m) < 1/k$. Also $\lim_{n\to\infty}\theta(x,x_n,Tx)$ and $\lim_{n\to\infty}\theta(x,x_n, x_{n+1})$ exist and are finite, then T has a unique fixed point.

3. A complex-valued new extended rectangular b-metric space

Here, we extend the concept of a complex-valued extended rectangular b- metric space to a complex valued new extended rectangular b- metric space.
We first define a new extended rectangular b-metric space.

Definition 3.1. *Let X be a non-empty set and $\theta : X \times X \times X \to [1,\infty)$ be a function. Then $d_\theta : X \times X \to [0,\infty)$ is called a new extended rectangular b-metric if for all $x, y, u, z \in X$, the following properties hold:*
(i) $d_\theta(x,y) \geq 0$ and $d_\theta(x, y) = 0$ iff $x = y$;
(ii) $d_\theta(x, y) = d_\theta(y, x)$;
(iii) $d_\theta(x, z) \leq \theta(x,y,z)\ [d_\theta(x, y) + d_\theta(y,u) + d_\theta(u,z)]$.
The pair (X, d_θ) is called a new extended rectangular b-metric space.

The complex-valued new extended rectangular b-metric space is defined as follows:

Definition 3.2. *Let X be a non-empty set and $\theta : X \times X \times X \to [1, \infty)$. Then $d_\theta : X \times X \to \mathbb{C}$ is called a complex-valued new extended rectangular b-metric if for all $x, y, u, z \in X$, the following properties hold:*
(i) $0 \preceq d_\theta(x, y)$ and $d_\theta(x, y) = 0$ iff $x = y$
(ii) $d_\theta(x, y) = d_\theta(y, x)$;
(iii) $d_\theta(x, z) \preceq \theta(x, y, z)[d_\theta(x, y) + d_\theta(y, u) + d_\theta(u, z)]$.
The pair (X, d_θ) is called a complex-valued new extended rectangular b-metric space.

Definition 3.3. *Let (X, d) be a complex-valued new extended rectangular b-metric space and $\theta : X \times X \times X \to [1, \infty)$ be a given function. Let $T : X \to X$ be such that*

$$d_\theta(Tx, Ty) \preceq f(M(x, y)), \forall x, y \in X, \tag{3}$$

where $f \in F$, $M(x, y) = max \{d_\theta(x, y), d_\theta(x, Tx), d_\theta(y, Ty)\}$ and F is the set of all continuous and non-decreasing functions $f : [0, \infty) \to [0, \infty)$ satisfying the following condition:

$$\lim_{n\to\infty} f^n(t) = 0, \forall t > 0.$$

Theorem 3.1. *Let (X, d_θ) be a complex-valued new extended rectangular b-metric space and $\theta : X \times X \times X \to [1, \infty)$. Let $T : X \to X$ satisfy (3). Suppose that for each $x_0 \in X$ and for each $t > 0$,*

$$\sup_{m\geq 1} \lim_{n\to\infty} \frac{f^{n+1}(t)}{f^n(t)} \theta(x_{n+1}, x_{n+2}, x_m) \prec 1$$

where $x_n = T^n(x_0)$ for $n \in \mathbb{N}$. Also, assume that for X,

$$\lim_{n\to\infty} \theta(x, x_n, x_{n+1}) \qquad and \qquad \lim_{n\to\infty} \theta(x, x_n, Tx)$$

exist and are finite. Then T has a unique fixed point, say $z \in X$. Also, $T^n y \to z$ for each $y \in X$.

Proof. Consider $x_n = T^n(x_0)$ for each $x_0 \in X$.
If for $n \in \mathbb{N}$, $x_n = x_{n+1} = Tx_n$ then x_n is a fixed point of T. Otherwise, assume that $x_n \neq x_{n+1}$. From (3), one writes

$$d_\theta(x_n, x_{n+1}) = d_\theta(Tx_{n-1}, Tx_n) \preceq fM(x_{n-1}, x_n),$$

where

$$\begin{aligned} M(x_{n-1}, x_n) &= max \{d_\theta(x_{n-1}, x_n), d_\theta(x_{n-1}, Tx_{n-1}), d_\theta(x_n, Tx_n)\} \\ &= max \{d_\theta(x_{n-1}, x_n), d_\theta(x_{n-1}, x_n), d_\theta(x_n, x_{n+1})\} \\ &= max \{d_\theta(x_{n-1}, x_n), d_\theta(x_n, x_{n+1})\}. \end{aligned}$$

If for some $n \in \mathbb{N}$,

$$M(x_{n-1}, x_n) = max \{d_\theta(x_{n-1}, x_n), d_\theta(x_n, x_{n+1})\} = d_\theta(x_n, x_{n+1})$$

then

$$0 \prec d_\theta(x_n, x_{n+1}) \preceq fM(x_n, x_{n+1}) \prec d_\theta(x_n, x_{n+1})$$

that is,

$$d_\theta(x_n,\ x_{n+1}) \prec d_\theta(x_n,\ x_{n+1})$$

which is a contradiction. Thus, for all $n \geq 1$,

$$M(x_{n-1},\ x_n) = max\ \{d_\theta(\ x_{n-1},\ x_n), d_\theta(\ x_n,\ x_{n+1})\} = d_\theta(\ x_{n-1},\ x_n)$$

and we have for all $n \geq 1$,

$$0 \prec d_\theta(\ x_n,\ x_{n+1}) \preceq fM(x_{n-1},\ x_n) \prec d_\theta(\ x_{n-1},\ x_n). \tag{4}$$

Continuing in this way we have for all $n \geq 0$,

$$0 \prec d_\theta(\ x_n,\ x_{n+1}) \prec f^n d_\theta(x_0,\ x_1) \tag{5}$$

Therefore, there exists $l \geq 0$ such that

$$\lim_{n\to\infty} d_\theta(x_n, x_{n+1}) = l.$$

Letting $n \to \infty$ in (4), we have

$$l \preceq f(l)$$

which holds unless $l = 0$. Thus,

$$\lim_{n\to\infty} d_\theta(x_n, x_{n+1}) = 0. \tag{6}$$

For $m > n$ we have

$$\begin{aligned}
d_\theta(x_n,\ x_m) &\preceq \theta(x_n,\ x_{n+1},\ x_m)[d_\theta(x_n,\ x_{n+1}) + d_\theta(x_{n+1},\ x_{n+2}) + d_\theta(x_{n+2},\ x_m)] \\
&\preceq \theta(x_n, x_{n+1}, x_m) d_\theta(x_n,\ x_{n+1}) + \theta(\ x_n, x_{n+1}, x_m)[d_\theta(x_{n+1},\ x_{n+2}) + d_\theta(x_{n+2}, x_m)] \\
&\preceq \theta(x_n,\ x_{n+1},\ x_m) d_\theta(x_n,\ x_{n+1}) + \theta(x_n,\ x_{n+1},\ x_m)\theta(x_{n+1}, x_{n+2}, x_m) d_\theta(x_{n+1},\ x_{n+2}) \\
&\quad +... + \theta\ (x_n,\ x_{n+1},\ x_m)\theta\ (x_{n+1},\ x_{n+2},\ x_m)...\theta\ (x_{m-2},\ x_{m-1},\ x_m) d_\theta(x_{m-1},\ x_m) \\
&\preceq \theta\ (x_n,\ x_{n+1},\ x_m) f^n d_\theta(x_0,\ x_1) + \theta\ (x_n,\ x_{n+1},\ x_m)\theta\ (x_{n+1},\ x_{n+2},\ x_m) f^{n+1} d_\theta(x_0,\ x_1) \\
&\quad +... + \theta\ (x_n,\ x_{n+1},\ x_m)\theta\ (x_{n+1},\ x_{n+2},\ x_m)...\theta\ (x_{m-2},\ x_{m-1},\ x_m) f^{m-1} d_\theta(x_0,\ x_1) \\
&\preceq \theta\ (x_1,\ x_2,\ x_m)\theta\ (x_2,\ x_3,\ x_m)...\theta\ (x_n,\ x_{n+1},\ x_m) f^n d_\theta(x_0,\ x_1) \\
&\quad + \theta(x_1,\ x_2,\ x_m)\theta\ (x_2,\ x_3,\ x_m)...\theta\ (x_n,\ x_{n+1},\ x_m)\theta(x_{n+1},\ x_{n+2},\ x_m) f^{n+1} d_\theta(x_0,\ x_1) \\
&\quad +... + \theta(x_1,\ x_2,\ x_m)\theta\ (x_2,\ x_3,\ x_m)...\theta\ (x_{m-2},\ x_{m-1},\ x_m) f^{m-1} d_\theta(x_0,\ x_1).
\end{aligned}$$

Let

$$S_n = \sum_{j=1}^{n} f^j(d_\theta(x_0, x_1))\Pi_{i=1}^{j}\theta(x_i,\ x_{i+1},\ x_m).$$

Thus for $m > n$, we have

$$d_\theta(x_n,\ x_m) \preceq [S_{m-1} - S_n].$$

Equivalently,

$$|d_\theta(x_n,\ x_m)| \leq |S_{m-1} - S_n|.$$

Consider the series

$$\sum_{n=1}^{\infty} f^n(d_\theta(\ x_0,\ x_1))\Pi_{i=1}^{n}\theta(x_i,\ x_{i+1},\ x_m).$$

Take $u_n = \sum_{n=1}^{\infty} f^n(d_\theta(x_0, x_1))\Pi_{i=1}^{n}\theta(x_i, x_{i+1}, x_m)$. We have

$$\frac{u_{n+1}}{u_n} = \frac{f^{n+1}(d_\theta(x_0, x_1))}{f^n(d_\theta(x_0, x_1))}\ \theta(x_{n+1}, x_{n+2}, x_m).$$

Since

$$\sup_{m\geq 1}\lim_{n\to\infty}\frac{f^{n+1}(t)}{f^n(t)}\theta(x_{n+1}, x_{n+2}, x_m) \prec 1$$

for each $m \in \mathbb{N}$, the series

$$\sum_{n=1}^{\infty} f^n(d_\theta(x_0,x_1))\Pi_{i=1}^{n}\theta(x_i,x_{i+1},x_m)$$

converges by ratio test. When $n \to \infty$, $S_n \longrightarrow 0$ or equivalently,

$$|d_\theta(x_n, x_m)| \longrightarrow 0 \tag{7}$$

which shows that $\{x_n\}$ is a Cauchy sequence in X. By completeness of X, there exists some $z \in X$ such that

$$\lim_{n\to\infty} x_n = z. \tag{8}$$

Now, to show that z is a fixed point of T, consider

$$\begin{aligned} d_\theta(z, Tz) &\preceq \theta(z,x_{n+1}, Tz)[d_\theta(z,x_{n+1}) + d_\theta(x_{n+1}, x_{n+2}) + d_\theta(x_{n+2},Tz)]\\ &\preceq \theta(z,x_{n+1}, Tx)[d_\theta(z,x_{n+1}) + d_\theta(x_{n+1}, x_{n+2}) + d_\theta(Tx_{n+1},Tz)]\\ &\preceq \theta(z,x_{n+1},Tx)[d_\theta(z, x_{n+1}) + d_\theta(x_{n+1}, x_{n+2}) + f\ max[d_\theta(x_{n+1},z),\\ &\quad d_\theta(x_{n+1},Tx_{n+1}), d_\theta(z,Tz)]].\end{aligned}$$

Using (6), (7) and (8) when $n \to \infty$,

$$d_\theta(z,Tz) \preceq 0.$$

Hence, $Tz = z$. To prove uniqueness, let z and z^* be two fixed points of T.
Now,

$$\begin{aligned} d_\theta(z, z^*) &= d_\theta(Tz, Tz^*)\\ &\preceq fM(z, z^*)\\ &= f[\max\ [d_\theta(z, z^*), d_\theta(z, Tz), d_\theta(z^*, Tz^*)]]\\ &= fd_\theta(z, z^*)\\ &\prec d_\theta(z\ ,z^*)\end{aligned}$$

which is a contradiction. Thus, T has a unique fixed point. □

We have the following corollary.

Corollary 3.1. *Let (X, d_θ) be a complex-valued extended rectangular b-metric space and $\theta : X \times X \to [1,\infty)$. Let $T : X \to X$ satisfy*

$$d_\theta(x, y) = k[max\ \{d_\theta(x, y), d_\theta(x, Tx), d_\theta(y,Ty)\}].$$

Suppose that for each $x_0 \in X$ and for each $t > 0$,

$$\sup_{m\geq 1}\lim_{n\to\infty}\theta(x_n, x_m) \prec 1/k$$

where $x_n = T^n(x_0)$ for $n \in \mathbb{N}$. Also, assume that for X,

$$\lim_{n\to\infty}\theta(x, x_n)$$

exists and is finite. Then T has a unique fixed point, say $z \in X$. Also $T^n y \to z$ for each $y \in X$.

4. A complex-valued new extended fuzzy rectangular b-metric space

In this section, we extended the complex valued extended fuzzy rectangular b-metric space to a complex valued new extended fuzzy rectangular b-metric space. Here, we present some of the basic notions which are helpful in defining and proving our results. We set $P = \{(a, b) : 0 \leq a < \infty, 0 \leq b < \infty\} \subset \mathbb{C}$. The elements $(0,0)$, $(1,1) \in P$ are denoted by ϕ and l respectively.
Define a partial ordering on $\mathbb{C}$ by $c_1 \preceq c_2$ if and only if $c_2 - c_1 \in P$.
$c_1 \prec c_2$ means $\mathrm{Re}(c_1) < \mathrm{Re}(c_2)$ and $\mathrm{Im}(c_1) < \mathrm{Im}(c_2)$.
We define the closed unit complex interval.

$$I = \{(a, b) : 0 \leq a < 1, 0 \leq b < 1\}$$

and open unit complex interval

$$I_\phi = \{(a, b) : 0 \leq a < 1, 0 \leq b < 1\}$$

and

$$P_\phi = \{(a, b) : 0 \leq a < \infty, 0 \leq b < \infty\}.$$

Definition 4.1. *[17] A binary operation $* : I \times I \to I$ is called a complex valued triangular norm (t-norm) if it satisfies the following conditions:*
(i) $c_1 * c_2 = c_2 * c_1$,
(ii) $c_1 * c_2 \preceq c_3 * c_4$, *whenever* $c_1 \preceq c_3$ *and* $c_2 \preceq c_4$;
(iii) $(c_1 * c_2) * c_3 = c_1 * (c_2 * c_3)$,
(iv) $c * \theta = \theta$ *and* $c * l = c$
for all $c, c_1, c_2, c_3, c_4 \in I$.

Example 4.1. *[17] The following are three basic complex valued t-norms:*
(i) The minimum t-norm, $T_M(c_1, c_2,) = (min\{a_1, a_2\}, min\{b_1, b_2\})$.
(ii) The product t-norm, $T_P(c_1, c_2) = (a_1a_2, b_1b_2)$.
(iii) The Lukasiewicz t-norm, $T_L(c_1, c_2) = (max\ \{a_1 + a_2 - 1, 0\}, max\ \{b_1 + b_2 - 1, 0\})$, *where* $c_1 = (a_1, b_1)$, $c_2 = (b_1, b_2) \in I$.

Definition 4.2. *[17] Let X be a nonempty set, $\star$ be a continuous complex-valued t-norm and M be a complex fuzzy set on $X \times X \times P_\phi$ if for all $x, y, z \in X$ and $c, c' \in P_\phi$, we have:*

(1bM): $\phi \prec M(x, y, c)$;

(2bM): $M(x, y, c) = l$, *for every* $c \in P_\phi \Leftrightarrow x = y$;

(3bM): $M(x, y, c) = M(y, x, c)$;

(4bM): $M(x, y, c) \star M_\theta(y, z, c') \preceq M(x, z, c + c')$;

(5bM): $M(x, y, \cdot) : P_\phi \to I$ *is left-continuous.*

Then the quadruple $(X, M, \star)$ is a complex-valued fuzzy metric space and M_θ is a complex valued fuzzy metric on X.

Definition 4.3. *[18] Let X be a nonempty set, $s \geq 1$ and $\star$ be a continuous complex-valued t-norm and M be a complex fuzzy set on $X \times X \times P_\phi$ if for all $x, y, z \in X$ and $c, c' \in P_\phi$, we have:*

(1bM): $\phi \prec M(x, y, c)$;

(2bM): $M(x, y, c) = l$, *for every* $c \in P_\phi \Leftrightarrow x = y$;

(3bM): $M(x,\ y,\ c)\ = M(y,\ x,\ c)$;

(4bM): $M(x,\ y,\ c) \star\ M(y,\ z,\ c') \preceq M(x,\ z,\ s(c+c'))$;

(5bM): $M(x,\ y, \cdot):\ P_{\phi} \to I$ *is left continuous.*

Then the quadruple $(X,\ M, \star, s)$ *is a complex-valued fuzzy b-metric space and* M *is a complex valued fuzzy b-metric on* X.

Now, the notion of a complex-valued new extended fuzzy rectangular b-metric space and some basic terminologies will be presented.

Definition 4.4. *Let* X *be a nonempty set ,* θ *be a function defined as* $\theta : X \times X \times X \to [1, \infty)$, $\star$ *be a continuous complex-valued t-norm and* M_{θ} *be a complex new extended fuzzy rectangular set on* $X \times X \times\ P_{\phi}$ *if for all* $x,\ y,\ z \in\ X$ *and* $c,\ c'\ \in\ P_{\phi}$, *we have:*

(1bM$_{\theta}$): $\phi \prec M_{\theta}(x,\ y,\ c)$;

(2bM$_{\theta}$): $M_{\theta}(x,\ y,\ c) = l$, *for every* $c \in\ P_{\phi} \Leftrightarrow x = y$;

(3bM$_{\theta}$): $M_{\theta}(x,\ y,\ c)\ = M_{\theta}(y,\ x,\ c)$;

(4bM$_{\theta}$): $M_{\theta}(x,\ y,\ c) \star M_{\theta}(y, u, c') \star M_{\theta}(u, z,\ c'') \preceq M_{\theta}(x, z, \theta(x,\ y,\ z)(c + c' + c''))$;

(5bM$_{\theta}$): $M_{\theta}(x, y, \cdot):\ P_{\phi} \to I$ *is left continuous .*

Then the quadruple $(X, M_{\theta}, \star, \theta)$ *is a complex-valued new extended fuzzy rectangular b-metric space and* M_{θ} *is a complex-valued new extended fuzzy rectangular b-metric on* X.

Remark 4.1. *(1): If* $\theta(x, y, z) = \theta(x,\ y)$, *then a complex-valued new extended fuzzy rectangular b-metric space becomes a complex-valued extended fuzzy rectangular b-metric space.*
(2): If $\theta(x,\ y),\ = s$ *then a complex-valued extended fuzzy rectangular b-metric space reduces to a complex-valued fuzzy rectangular b-metric space.*
(3): If $s = 1$, *then a complex-valued fuzzy rectangular b-metric space reduces to a complex-valued fuzzy rectangular*

Example 4.2. *Let* $(X,\ d_{\theta}, \theta)$ *be a new extended b-metric space. Consider* $M_{d_{\theta}} : X \times\ X \times\ P_{\phi} \to I$ *such that*

$$M_{d_{\theta}}(x,\ y,\ c) = \frac{a.\ b}{ab + d_{\theta}(x,\ y)} l,$$

where $c = (a, b) \in\ P_{\phi}$ *and* $*$ *is a min t-norm. Then* $(X, M_{d_{\theta}}, *, \theta)$ *is a complex valued new extended fuzzy rectangular b-metric space.* $M_{d_{\theta}}$ *is known as a standard complex-valued new extended fuzzy rectangular b-metric .*

Remark 4.2. *Every complex-valued new extended fuzzy rectangular b-metric may not be induced by a new extended b-metric space .*

Example 4.3. *Let* $X\ = (+3, \infty)$ *and let* $M_{d_{\theta}} : X \times X \times\ P_{\phi} \to I$ *be defined by*

$$M_{d_{\theta}}(x,\ y,\ t) = \begin{cases} (\frac{1}{x} + \frac{1}{y})l & if \quad x\ \neq y; \\ l, & if \quad x\ =\ y \end{cases}$$

where $c\ =\ (a,\ b) \in\ P_{\phi}$ *and* $*$ *is a maximum complex valued t-norm, then* $(X, M_{d_{\theta}}, *, \theta)$ *is a complex-valued new extended fuzzy rectangular b-metric space.*
It can be easily seen that there is not a new extended b-metric d *on* X *inducing the given complex valued new extended fuzzy rectangular b-metric space .*

Definition 4.5. *Given a function $\theta : X\times X\times X \to [1,\infty)$. A function $f : \mathbb{R} \to \mathbb{R}$ is known as $\theta-$nondecreasing if for any $t < u$ we have*

$$f(t) \leq f(\theta(x, y, z)u).$$

Lemma 4.1. *Let $(X, M_{d_\theta}, *, \theta)$ be a complex valued new extended fuzzy rectangular b-metric space. For all $x, y \in X$ and $c_1, c_2 \in \mathbb{C}$, the mapping $M_\theta(x, y, \cdot) : [0,\infty) \to [0, 1]$ is $\theta-$non-decreasing. That is, if $c_1 \prec c_2$, then $M_\theta(x, y, c_1) \preceq M_\theta(x, y, \theta(x, x, y)c_2)$.*

Proof. For $c_1 \prec c_2$, where $c_1, c_2 \in P_\phi$, and $c_2 - c_1 \in P_\phi$, we have

$$M_\theta(x, y, c_1) = l*M_\theta(x, y, c_1)*l = M_\theta(x, x, \frac{c_2 - c_1}{2})*M_\theta(x, y, c_1)*M_\theta(y, y, \frac{c_2 - c_1}{2}) \preceq M_\theta(x, y, \theta(x, x, y)c_2)$$

that is,

$$M_\theta(x, y, c_1) \preceq M_\theta(x, y, \theta(x, x, y)c_2).$$

□

Definition 4.6. *Let $(X, M_\theta, *, \theta)$ be a complex-valued new extended fuzzy rectangular b-metric space.*
(1) A sequence $\{x_n\}$ in X converges to $x \in X$ if for every $r \in I_\phi$ and every $c \in P_\phi$, there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$,

$$l - r \prec M_\theta(x_n, x, c).$$

Equivalently,

$$\lim_{n\to\infty} x_n = x.$$

(2) A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if for every $c \in P_\phi$,

$$\lim_{n\to\infty} \inf_{n>m} M_\theta(x_n, x_m, c) = l.$$

(3) $(X, M_\theta, *, \theta)$ is said to be a complete complex-valued new extended fuzzy rectangular b-metric space if for every Cauchy sequence x_n in $(X, M_\theta, *, \theta)$, there exists $x \in X$ such that

$$\lim_{n\to\infty} x_n = x.$$

(4) An open ball with centre at $x \in X$, radius $r \in I_\phi$ and $c \in P_\phi$ is defined as

$$B(x, r, c) = \{y \in X : l - r \prec M_\theta(x, y, c)\}.$$

(5) Let $(X, M_\theta, \star, \theta)$ be a complex-valued new extended fuzzy rectangular b-metric space and τ be defined as :

$$\tau = \{A \subset X : x \in A \ iff \ \exists\, r \in I_\phi \text{ and } c \in P_\phi \text{ such that } B(x, r, c) \subset A\}.$$

Then τ is a topology on X.

We prove the Banach contraction theorem in complex-valued new extended fuzzy rectangular b-metric spaces.

Theorem 4.1. ***(Banach contraction theorem):*** *Let $(X, M_\theta, \star, \theta)$ be a complete complex-valued new extended fuzzy rectangular b-metric space such that for every sequence $\{c_n\} \in P_\phi$ with $\lim_{n\to\infty} c_n = \infty$ we have*

$$\lim_{n\to\infty} \inf_{y\in X} M_\theta(x, y, c_n) = l \ \forall \ x \in X.$$

Let $T : X \to X$ be a mapping satisfying

$$M_\theta(Tx, Ty, \frac{kc}{\theta(x, x, y)}) \succeq M_\theta(x, y, c), \ \forall\, x, y \in X \tag{9}$$

where $c \in P_\phi$ and $0 < k < 1$, then there is a unique fixed point of T.

Proof. Let $x_0 \in X$ be an arbitrary element. Let x_n be in X so that

$$x_n = Tx_{n-1} \ , (n \ \in \ \mathbb{N}).$$

If $x_n \ = x_{n-1}$, then x_n is a fixed point of T.
Let $x_n \ \neq x_{n-1}$ for all $n \ \in \ \mathbb{N}$.
Now, we will prove that $\{x_n\}$ is a Cauchy sequence in X. Define

$$B_n = \{M_\theta(x_n, \ x_m, c) : m \ > \ n\}$$

for all $n \in \ \mathbb{N}$ and $c \in P_\phi$.
Since

$$\theta \prec M_\theta(x_n, \ x_m, \ c) \prec \ l.$$

and $\inf B_n \ = \beta_n$ exists for all $n \in \ \mathbb{N}$, we have

$$M_\theta(x_n, \ x_m, \ c) \preceq M_\theta(x_n, \ x_m, \ \frac{\theta(x_n, \ x_n, \ x_m)c}{k}) \preceq M_\theta(Tx_n, \ Tx_m, \ c) \preceq M_\theta(x_{n+1}, x_{m+1}, c). \tag{10}$$

Thus, from

$$\theta \preceq \beta_n \preceq \beta_m \preceq l,$$

it follows that $\{\beta_n\}$ is a monotonic sequence in P. Therefore, we have an $l_0 \ \in \ P$ satisfying

$$\lim_{n\to\infty} \beta_n \ = \ l_0 \ .$$

Now,

$$\begin{aligned}
M_\theta(x_{n+1}, \ x_{m+1}, c) \ &\succeq \ M_\theta(x_n, \ x_m, \frac{\theta(x_n, \ x_n, \ x_m)c}{k}) \\
&= \ M_\theta(Tx_{n-1}, \ Tx_{m-1}, \frac{\theta(x_n, x_n, x_m)c}{k}) \\
&\succeq \ M_\theta(x_{n-1}, \ x_{m-1}, \frac{\theta(x_n x_n, x_m)\theta(x_{n-1}, x_{n-1}, x_{m-1})c}{k^2}) \\
&= \ M_\theta(Tx_{n-2}, Tx_{m-2}, \frac{\theta(x_n x_n, x_m)\theta(x_{n-1}, x_{n-1}, x_{m-1})c}{k^2}) \\
&\succeq \ M_\theta(x_{n-2}, x_{m-2}, \frac{\theta(x_n x_n, x_m)\theta(x_{n-1}, x_{n-1}, x_{m-1})\theta(x_{n-2}, x_{n-2}, x_{m-2})c}{k^3}) \\
&\ldots \\
&\succeq \ M_\theta(x_0, \ x_{m-n}, \frac{\prod_{i,j=1}^{n,m} \theta(x_i, x_i, x_j)c}{k^{n+1}}).
\end{aligned}$$

Thus,

$$\begin{aligned}
\beta_{n+1} \ &= \ \inf_{m \ > \ n} M_\theta(x_{n+1}, \ x_{m+1}, \ c) \\
&\succeq \ \inf_{m>n} M_\theta(x_0, x_{m-n}, \frac{\prod_{i, \ j \ =1}^{n,m} \theta(x_i, x_i, x_j)c}{k^{n+1}}) \\
&\succeq \ \inf_{y\in X} M_\theta(x_0, y, \frac{\prod_{i, \ j \ =1}^{n,m} \theta(x_i, x_i, x_j)c}{k^{n+1}}).
\end{aligned}$$

Since $\lim_{n\to\infty}\frac{\prod_{i,j=1}^{n,m}\theta(x_i,x_i,x_j)c}{k^{n+1}}=\infty$, we obtain

$$l_0 \succeq \lim_{n\to\infty}\inf_{y\in X} M_\theta(x_0,y,\frac{\prod_{i,j=1}^{n,m}\theta(x_i,x_i,x_j)c}{k^{n+1}}) \;=\; l$$

which implies that $l_0 \;=\; l$.
Thus, $\{x_n\}$ is a Cauchy sequence in X. Since X is a complete complex valued new extended fuzzy rectangular b-metric space, there exists a point $x\in X$ such that for all $c\in P_\phi$,

$$\lim_{n\to\infty} M_\theta(x_n,x,c)=l.$$

Next, we will show that x is a fixed point of T. Now by using $(4bM_\theta)$, one writes

$$\begin{aligned} M_\theta(x,\ Tx,c) &\succeq M_\theta(x,\ x_{n+1},\frac{c}{3\theta(x,\ x_{n+1},\ Tx)})\star M_\theta(x_{n+1},\ x_{n+2},\frac{c}{3\theta(x,\ x_{n+1},\ Tx)})\\ &\quad\star M_\theta(x_{n+2},Tx,\frac{c}{3\theta(x,x_{n+1},Tx)})\\ &= M_\theta(x,\ x_{n+1},\frac{c}{3\theta(x,\ x_{n+1},\ Tx)})\star M_\theta(Tx_n,\ Tx_{n+1},\frac{c}{3\theta(x,x_{n+1},\ Tx)})\\ &\quad\star M_\theta(Tx_{n+1},\ Tx,\frac{c}{3\theta(x,\ x_{n+1},\ Tx)})\\ &\succeq M_\theta(x,\ x_{n+1},c/3k)\star M_\theta(x_n,x_{n+1},c/3k)\star M_\theta(x_{n+1},x,\ c/3k). \end{aligned}$$

When $n\to\infty$, we have

$$\begin{aligned} M_\theta(x,Tx,c) &\succeq l\star l\star l=l\\ &\implies Tx=x. \end{aligned}$$

Uniqueness: Let x and $x^\star$be two fixed points of the mapping T. So $Tx \;=\; x$ and $Tx^\star=x^\star$, that is, $M_\theta(Tx,\ x,\ c)=l$ and $M_\theta(Tx^\star,x^\star,\ c)=l$. Now,

$$\begin{aligned} M_\theta(x,\ x^\star,\ c) &= M_\theta(Tx,\ Tx^\star,\ c)\\ &\succeq M_\theta(x,x^\star,\frac{\theta(x,\ x,x^\star)c}{k}) \end{aligned}$$

which is a contradiction. Hence, $x \;=\; x^\star$. □

Now, we give an example to make ou result meaningful.

Example 4.4. *Let* $X \;=\; [0,1]$ *and consider* $M_\theta: X\times X\times\ P_\phi\to I$ *such that*

$$M_\theta(x,\ y,c)=\frac{a.\ b}{ab+(x\ \ -y)^2}l,$$

where $c \;=(a,\ b)\in\ P_\phi$ *then* $(X,M_\theta,\star,\theta)$ *is a complete complex-valued new extended fuzzy rectangular b-metric space with* $\theta=2$ *and for any sequence* $\{c_n\}$ *in* P_ϕ *with* $\lim_{n\to\infty}c_n=\infty$*, we have*

$$\lim_{n\to\infty}\inf_{y\in X} M_\theta(x,y,c_n)=l\ \forall\ x\in\ \ X.$$

We define a mapping $T:\ \ X\to\ \ X$ *as*

$$Tx=\alpha x^2$$

where $\alpha\in(0,\frac{1}{4})$*. It is easy to see that*

$$M_\theta(Tx,\ Ty,\frac{kc}{2})\succeq M_\theta(x,\ y,c),\ \forall\ x,\ y\in X \tag{11}$$

where $c \in P_\phi$ *and* $k = 4\alpha \in (0,1)$. *Hence, all the conditions of above theorem are satisfied and 0 is the unique fixed point of T.*

Next, we establish a common fixed point theorem in the setting of a complex-valued new extended fuzzy rectangular b-metric space.

Theorem 4.2. *Let* $(X, M_\theta, \star, \theta)$ *be a complete complex-valued new extended fuzzy rectangular b-metric space such that for every sequence* $\{c_n\} \in P_\phi$ *with* $\lim_{n\to\infty} c_n = \infty$ *we have*

$$\lim_{n\to\infty} \inf_{y\in X} M_\theta(x, y, c_n) = l, \ \forall \ x \in X$$

$T, h : X \to X$ *be two self mappings satisfying the following conditions:*

1. $TX \subseteq hX$,
2. T and h commute on X,
3. h is continuous on X,
4. $M_\theta(Tx, Ty, \frac{kc}{\theta}) \succeq (M_\theta(hx, hy, c)$ for all $x, y \in X$ and $c \in P_\phi$, where $k \in (0,1)$.

Then T and h have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be an arbitrary point and a sequence $\{x_n\}$ is defined. Since $TX \subseteq hX$, there exists $x_1 \in X$ such that

$$y_1 = hx_1 = Tx_0.$$

Similarly, there exists $x_2 \in X$ such that

$$\begin{aligned} y_2 &= hx_2 = Tx_1 \\ \text{and} \quad hx_3 &= Tx_2. \end{aligned}$$

In general for all $n \in \mathbb{N}$,

$$y_n = hx_n = Tx_{n-1}.$$

Also assume that $y_n \neq y_{n+1}$. Otherwise, T and h have a a coincidence point.
Now, we will show that $\{y_n\}$ is a Cauchy sequence.
For all $n \in \mathbb{N}$ and $c \in P_\phi$, we define

$$B_n = \{M_\theta(y_n, y_m, c) : m > n\}$$

for all $n \in \mathbb{N}$ and $c \in P_\phi$.
Since

$$\phi \prec M_\theta(y_n, y_m, c) \prec l,$$

and $\inf B_n = \beta_n$ exists for all $n \in \mathbb{N}$, we have

$$\begin{aligned} M_\theta(y_n, y_m, c) &= M_\theta(hx_n, hx_m, c) \\ &\preceq M_\theta(hx_n, hx_m, \frac{\theta(y_n, y_n, y_m)c}{k}) \\ &\preceq M_\theta(Tx_n, Tx_m, c) \\ &\preceq M_\theta(hx_{n+1}, hx_{m+1}, c) = M_\theta(y_{n+1}, y_{m+1}, c). \end{aligned}$$

Recall that

$$\theta \preceq \beta_n \preceq \beta_{n+1} \preceq l.$$

It follows that $\{\beta_n\}$ is a monotonic sequence in P. Therefore, we have $l_0 \in P$ satisfying

$$\lim_{n\to\infty} \beta_n = l_0.$$

Now,

$$\begin{aligned}
M_\theta(y_{n+1}, y_{m+1}, c) &= M_\theta(hx_{n+1}, hx_{m+1}, c)\\
&= M_\theta(Tx_n, Tx_m, c)\\
&\succeq M_\theta(y_n, y_m, \frac{\theta(y_n, y_n, y_m)c}{k})\\
&= M_\theta(hx_n, hx_m, \frac{\theta(y_n, y_n, y_m)c}{k})\\
&= M_\theta(Tx_{n-1}, Tx_{m-1}, \frac{\theta(y_n, y_n, y_m)c}{k})\\
&\succeq M_\theta(y_{n-1}, \ y_{m-1}, \frac{\theta(y_n, \ y_n, \ y_m)\theta(y_{n-1}, y_{n-1}, y_{m-1})c}{k^2})\\
&= M_\theta(hx_{n-1}, hx_{m-1}, c)\frac{\theta(y_n, y_n, \ y_m)\theta(y_{n-1}, y_{n-1}, y_{m-1})c}{k^2}\\
&= M_\theta(Tx_{n-2}, Tx_{m-2}, \frac{\theta(y_n, y_n, \ y_m)\theta(y_{n-1}, \ y_{n-1}, \ y_{m-1})c}{k^2})\\
&\succeq M_\theta(y_{n-2}, \ y_{m-2}, \frac{\theta(y_n, \ y_n, \ y_m)\theta(y_{n-1}, \ y_{n-1}, \ y_{m-1})\theta(y_{n-2}, \ y_{n-2}, \ y_{m-2})c}{k^3})\\
&\cdots\\
&\succeq M_\theta(y_0, y_{m-n}, \frac{\prod_{i,j=1}^{n,m}\theta(y_i, y_i, y_j)c}{k^{n+1}})\\
&= M_\theta(hx_0, hx_{m-n}, \frac{\prod_{i,j=1}^{n,m}\theta(y_i, y_i, y_j)c}{k^{n+1}}).
\end{aligned}$$

Thus,

$$\begin{aligned}
\beta_{n+1} &= \inf_{m>n} M_\theta(hx_{n+1}, hx_{m+1}, c)\\
&\succeq \inf_{m>n} M_\theta(hx_0, hx_{m-n}, \frac{\prod_{i,j=1}^{n,m}\theta(y_i, y_i, y_j)c}{k^{n+1}})\\
&\succeq \inf_{y\in X} M_\theta(hx_0, y, \frac{\prod_{i,j=1}^{n,m}\theta(y_i, y_i, y_j)c}{k^{n+1}}).
\end{aligned}$$

Since $\lim_{n\to\infty} \frac{\prod_{i,j=1}^{n,m}\theta(y_i, y_i, y_j)c}{k^{n+1}} = \infty$ and by using equation (3), we obtain,

$$l_0 \succeq \lim_{n\to\infty}\inf_{y\in X} M_\theta(y_0, y, \frac{\prod_{i,j=1}^{n,m}\theta(y_i, y_i, y_j)c}{k^{n+1}}) = l$$

which implies that $l_0 = l$. Thus $\{y_n\} = \{hx_n\}$ is a Cauchy sequence in X.
Since X is a complete complex valued new extended fuzzy rectangular b-metric space, there exists a point $x \in X$ such that for all $c \in P_\phi$,

$$\lim_{n\to\infty} hx_n = x.$$

The continuity of h implies the continuity of T, and so

$$\lim_{n\to\infty} Thx_n = Tx.$$

Since T and h commute on X, we have

$$\lim_{n\to\infty} hTx_n = Tx.$$

Also, we know that

$$\lim_{n\to\infty} Tx_{n-1} = x$$

so this implies

$$\lim_{n\to\infty} hTx_{n-1} = hx.$$

Due to the uniqueness of the limit, we have $Tx = hx$. Thus, $hTx = TTx$.
By condition (4), we have

$$\begin{aligned} M_\theta(Tx,\ TTx,\ c) &\succeq M_\theta(hx, hTx, \frac{\theta(hx,hx,hTx)c}{k}) \\ &= M_\theta(Tx,\ TTx,\ \frac{\theta(hx,hx,hTx)c}{k}) \\ &\succeq \\ \ldots \\ &\succeq M_\theta(Tx, TTx, \frac{\theta^n(hx,hx,hTx)c}{k^n}) \\ &= M_\theta(Tx,\ hTx, \frac{\theta^n(hx,hx,hTx)c}{k^n}) \\ &\succeq \inf_{y\in X} M_\theta(Tx, y, \frac{\theta^n(hx,hx,hTx)c}{k^n}). \end{aligned}$$

As $n \to \infty$, we have

$$M_\theta(Tx, TTx, c) = l$$

$$\implies TTx = hTx = Tx.$$

That is, Tx is a common fixed point of h and T.
Uniqueness: Let Tx and $x^\star$be two common fixed points of the mappings T and h, so using the condition (4) with $x = Tx$ and $y = x^\star$ we have

$$\begin{aligned} l &\succeq M_\theta(Tx,\ x^\star, c) \\ &= M_\theta(TTx,\ Tx^\star, c) \\ &\succeq M_\theta(hTx, hx^\star, \frac{\theta(x,\ x, y)c}{k}) \\ &= M_\theta(Tx,\ x^\star, \frac{\theta(x,\ x,\ y)c}{k}) \\ \ldots. \\ &\succeq M_\theta(Tx, x^\star, \frac{\theta^n(x,x,y)c}{k^n}) \\ &\succeq \inf_{y\in X} M_\theta(Tx,\ y, \frac{\theta^n(x,x,\ y)c}{k^n}). \end{aligned}$$

Now,

$$\lim_{n\to\infty} \frac{\theta^n(x,\ x,\ y)c}{k^n} = \infty,$$

and we have

$$M_\theta(Tx,\ x^\star, c) = l.$$

Thus, $Tx = x^\star$. It completes the proof. □

5. Applications to the existence of solutions of integral equations

In this section, we study the existence of a solution of the following integral equation by using our main results.

Theorem 5.1. *The given integral equation has one and only one solution in* $\mathbb{C}([0,\ 1],\mathbb{R}\ )$ *if the following four conditions are satisfied.*

$$x(t) = v(t) + \beta \int_0^1 \psi(c,\theta) f(\theta, x(\theta)) d\theta . \forall c \in [0,1] \tag{12}$$

1. $v : [0,1] \to \mathbb{R}$ is continuous ,
2. $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous, $f(c,x) \ \geq 0$ and there is $k \ \in \ [0,1)$ suc that
$$|f(c,x) \ - f(c,y)| \leq \ k|x \ - y|$$
for all $x,\ y \ \in \mathbb{R}$,
3. $\psi\ :\ [0,1] \times\ [0,1] \to \mathbb{R}$ is continuous at $c \in [0,1]$, for all $\theta \in [0,1]$ and measurable at $\theta \in [0,1]$ for all $c \in [0,1]$. Also $\psi(c,\theta) \geq\ 0$ and $\int_0^1 \psi(c,\theta) \leq L$,
4. $k^2L^2\beta^2 \leq 1/2$.

Proof. Let $X \ = \mathbb{C}([0,1\ ],\mathbb{R})$ and define a mapping $T : X \to \ X$ by

$$Tx(t) = v(t) + \beta \int_0^1 \psi(c,\theta) f(\theta, x(\theta)) d\theta, \quad c \in [0,1].$$

Define a mapping $M : X^2 \ \times P_\theta \ \to I$ by

$$M(\ x,\ y,c) = l - \sup_{t\in\ [0,1]} \frac{(x(t) \ - y(t))^2}{e^{ab}} l$$

where $c = (a,b) \in P_\phi$. Clearly, X is a complete complex valued new extended fuzzy rectangular b-metric space. Moreover, for all $x, y \in X$ and $t \in [0,1]$ we have

$$\begin{aligned}
|Tx(t) \ - \ Ty(t)| &= \beta|\int_0^1 \psi(c,\ \theta) f(\ \theta,\ x(\theta)) d\theta - \psi(c,\theta) f(\ \theta, y(\theta)) d\theta| \\
&\leq \beta \int_0^1 \psi(c,\theta)|f(\ \theta,\ x(\theta)) \ - f(\ \theta,\ y(\theta))| d\theta \\
&\leq \beta \int_0^1 \psi(c,\theta) k|x(\theta)) - y(\theta))| d\theta \\
&\leq \beta L k \sup_{t\in[0,1]} |x(t) \ - \ y(t)|.
\end{aligned}$$

From

$$\sup_{t\in\ [0,1]} |Tx(t) \ - \ Ty(t)| \leq \ \beta L k \sup_{t\in\ [0,1]} |x(t) \ - \ y(t)|$$

it follows that

$$\begin{aligned}
\sup_{t\in\ [0,1]} \frac{|Tx(t) \ - Ty(t)|^2}{e^{ab}} &\leq \beta^2 L^2 k^2 \sup_{t\in\ [0,1]} \frac{|x(t) \ - y(t)|^2}{e^{ab}} \\
&\leq \frac{1}{2} \sup_{t\in[0,1]} \frac{|x(t) \ - \ y(t)|^2}{e^{ab}}.
\end{aligned}$$

Thus, the integral equation has a unique solution in $X = \mathbb{C}([0,1],\mathbb{R})$. □

Next, we give an example of an integral equation and establish the existence of its solutions by using above theorem .

Example 5.1. *Consider the following integral equation*

$$x(t) \quad = \quad \frac{1}{1+t} + 2\int_0^1 \frac{\theta^2}{t^2+2}.\frac{|cosx(\theta)|}{5e^\theta}d\theta.$$

Here,

$$\beta = 2, v(t) = \frac{1}{1+t}, \psi(c,\theta) = \frac{\theta^2}{t^2+2}, \;\; f(t, \;\; x) = \frac{|cosx|}{5e^t}$$

Clearly, f is continuous on $[0, \;\; 1] \times \;\; \mathbb{R}$ *and*

$$\begin{aligned} |f(t, \;\; x) - f(t, \;\; y)| \quad &= \quad \frac{1}{5e^t}||cosx| \;\; - \;\; |cosy|| \\ &\leq \quad \frac{1}{5e^t}|cosx \;\; - \;\; cosy| \\ &\leq \quad \frac{1}{5}|cosx \;\; - \;\; cosy| \\ &\leq \quad \frac{1}{5}|x \;\; - \;\; y| \end{aligned}$$

for all $x, y \;\; \in \;\; \mathbb{R}$. *f satisfies the condition (2) of the above theorem with* $k \;\; = \;\; 1/5$. *Also,* v *is continuous. We have*

$$\int_0^1 \psi(c,\theta)d\theta = \int_0^1 \frac{\theta^2}{t^2+2}d\theta = \frac{1}{t^2+2}.\frac{1}{3} \leq \frac{1}{6} = L.$$

Moreover,

$$k^2 L^2 \beta^2 \leq 1/2.$$

Since all the conditions of above theorem are fulfilled, the given integral equation has a unique solution in $X \;\; = \;\; \mathbb{C}([0, \;\; 1], \mathbb{R})$.

Data Availability

No data is associated with this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' contributions

All authors contributed equally and significantly in writing this article.

References

[1] B.C. Dhage, Condensing mappings and applications to existence theorems for the common solution of differential equations, Bull. Korean Math. Soc. 36 (3) (1999), 565-578.
[2] H. Pathak, B. Fisher, Common fixed point theorems with applications in dynamic programming, Glas. Mat. 31 (1996), 321-328.
[3] H. Pathak, S. Mishra, A. Kalinde, Common fixed point theorems with applications to non-linear integral equations, Demonstratio Math. 32 (3) (1999), 547-564.
[4] L.A. Zadeh, Information and control. Fuzzy sets, 8(3) (1965), 338–353.
[5] I. Kramosil, J. Michálek, Fuzzy metrics and statistical metric spaces, Kybernetika, 11 (5) (1975), 336–344.

[6] A. George, P. Veeramani, On some results in fuzzy metric spaces. Fuzzy sets and systems, 64(3) (1994), 395–39.
[7] S. Heilpern, Fuzzy mappings and fixed point theorem. Journal of Mathematical Analysis and Applications, 83(2) (1981), 566–569.
[8] M. Grabiec, Fixed points in fuzzy metric spaces. Fuzzy sets and systems, 27(3) (1988), 385–389.
[9] I.A. Bakhtin, The contraction mapping priquasi-metricuasi metric spaces. Func. An., Gos. Ped. Inst. Unianowsk, 30 (1980), 26–37.
[10] S. Czerwik, Contraction mappings in b-metric spaces. Acta Mathematica et Informatica Universitatis Ostraviensis, 1(1) (1993), 5–11.
[11] S. Czerwik, Nonlinear set-valued contraction mappings in b-metric spaces. Atti Del Seminario Matematico E Fisico Universita Di Modena, 46 (1998), 263–276.
[12] M.A. Khamsi, Remarks on cone metric spaces and fixed point theorems of contractive mappings, Fixed Point Theory Appl. 1 (2010), 315–398.
[13] M.A. Khamsi, N. Hussain, KKM mappings in metric-type spaces, Nonlinear Anal. 73(9) (2010), 3123–3129.
[14] A.C. Ran, M.C. Reurings, A fixed point theorem in partially ordered sets and some applications to matrix equations, proceedings of the American Mathematical Society, 132 (5) (2004), 1435–1443.
[15] S. Nădăban, Fuzzy b-Metric Spaces. International Journal of Computers Communications Control, 11(2) (2006), 273–281.
[16] A. Azam, B. Fisher, M. Khan, Common fixed point theorems in complex valued metric spaces, Numer. Funct. Anal. Optim. 32 (3) (2011), 243-253.
[17] S. Shukla, R. Rodriguez-Lopez, M. Abbas, Fixed point results for contractive mappings in complex-valued fuzzy metric spaces, Fixed Point Theory, 19 (2) (2018), 751-774.
[18] I. Demir, Fixed point theorems in complex valued fuzzy b-metric spaces with application to integral equations, Miskolc Mathematical Notes, 22 (1) (2021), 153–171.
[19] A. Mukheimer, Some common fixed point theorems in complex valued b-metric spaces, The Scientific World Journal, Volume 2014, Article ID 587825.
[20] T. Kamran, M. Samreen, Q. UL Ain, A generalization of b-metric space and some fixed point theorems. Mathematics, 5(2) (2017), 19.
[21] N. Ullah, M.S. Shagari, A. Azam, Fixed Point Theorems in Complex Valued Extended b-Metric Spaces, Moroccan Journal of Pure and Applied Analysis, 5(2) (2019), 140-163.
[22] Z. Mustafa, V. Parvaneh, M.M.M. Jaradat, Z. Kadelburg, Extended Rectangular b- Metric Spaces and Some Fixed Point Theorems for Contractive Mappings, Symmetry, 11 (2019), 594,
[23] M. Asim, M. Imdad, S. Radenovi c, Fixed Point Results in Extended Rectangular b - Metric Spaces with an Application, U.P.B. Sci. Bull., Series A, 81 (2) (2019).
[24] N. Ullah, M.S. Shagari, Fixed Point Results in Complex Valued Rectangular Extended *b*-Metric Spaces with Applications December 2020Mathematical Analysis and Convex Optimization, 1(2) (2020), 107–120.