Advances in the Theory of Nonlinear Analysis and its Applications 7 (2023) No. 5, 130-140.
https://doi.org/10.17762/atnaa.v7.i5.330
Available online at https://atnaea.org/
Research Article

Advances in the Theory of Nonlinear Analysis and its Applications
ISSN: 2587-2648
Peer-Reviewed Scientific Journal

Direct method of solving nonlinear ordinary differential equations through known functions

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Abstract

In this work, we develop a systematic procedure to obtain the general solutions of a class of nonlinear ordinary differential equations (ODEs) directly through the special functions or other known functions. By introducing a suitable transformation in the state variable/dependent variable of the given nonlinear ODE, we can relate it to one of the the special function equations, including Hermite's equation, Legendre's equation, and Laguerre's equation or other equations solvable through known functions, including isochronous and limit cycle solutions. This procedure can be further generalized to higher order nonlinear ODEs. Obtaining the general solutions of the nonlinear ODEs with the help of special functions is new to the literature to our knowledge.

Keywords: Nonlinear ordinary differential equations Special functions Hermite equation Legendre's equation Laguerre's equation.
2010 MSC: 68Q25, 68R10, 68U05.

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Received September 24, 2023; Accepted: November 27, 2023; Online: December 28, 2023.

1. Introduction

Finding the general solution of a given nonlinear ordinary differential equation (ODE) is still a difficult task. To derive the solutions of a given ODE, the Norwegian mathematician Sophus Lie introduced a powerful method now known as the Lie group method [1, 2, 3]. This method provides infinitesimal symmetries associated with the given ODE and, in general, these infinitesimal symmetries depend only on the dependent and independent variables of the given differential equation. From the underlying Lie point transformation, one may find the solution through order reduction procedure or construct the associated integrals of motion. However, not all nonlinear ODEs, in general, admit Lie point symmetries. Hence to prove the integrability of the given dynamical system few other methods have also been developed in the literature. Some of the familiar methods are Prelle-Singer procedure, the method of Darboux polynomials, Jacobi last multiplier procedure, λ-symmetries method and so on [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]. Each method/procedure has its own merits and demerits. In the literature the aforementioned methods have been applied to a wide range of problems.

Apart from the above, one may also solve the given nonlinear ODE by transforming it into a linear ODE or even to a known integrable/solvable nonlinear ODE. The transformation may be either local or nonlocal. Identifying a suitable transformation for the given equation is not a trivial task [15, 16, 17, 18, 19, 20]. In this work, we present a hybrid method to integrate nonlinear ODEs. In this integration technique, we bring together the method of linearization and direct integration. We assume the solution of the nonlinear ODE to exist in a rational form, that is, the ratio of two different functions in which the numerator is the one of special functions, for example the Hermite, Legendre and Laguerre function, or any other known functions and the denominator is the solution of a first-order ODE. Imposing the compatibility between this solution and the given equation, we fix the form of the nonlinear ODE. To derive the solution of the nonlinear ODE, we again consider the solution of the linear or solvable nonlinear ODEs, namely an arbitrarily chosen function which we introduce in the process. In our work, as examples, we consider the form of the linear ODEs as linear harmonic oscillator, damped harmonic oscillator, Hermite, Legendre and Laguerre equations. However, one may consider any other linear function also. As far as our knowledge goes, this is the first time in the literature, the solution of the nonlinear ODEs are obtained through such special functions.

The structure of the paper is as follows: In Sec 2, we present the procedure and in Sec 3, we consider four examples corresponding to second order ODEs and illustrate the method of deriving the solutions of nonlinear ODEs through known functions, including the special functions. Next, in Sec. 4, we indicate the extension of the method to third order ODEs. Finally, we present our conclusion in Sec.5.

2. Method: Second order nonlinear ODEs

Let us consider a second order nonlinear ODE in the form

$$\ddot{x} = P(t, x, \dot{x}), \tag{1}$$

where dot denotes differentiation with respect to time and P is an analytic function in t, x and $\dot{x}$. Let us assume that the ODE (1) admits a solution which is a ratio of two functions, that is

$$x(t) = w(t)/s(t), \tag{2}$$

where w is the solution of the differential equation

$$\ddot{w} = Q(t, w, \dot{w}), \tag{3}$$

and the function $s(t)$ satisfies the first order differential equation

$$\dot{s} = h(x, t)s. \tag{4}$$

One can note that most of the integrable nonlinear equations admit solutions of the form (2). In Eq. (4), $h(x,t)$ is a function which has to be chosen appropriately. The ODE (3) may be of any form but satisfies a homogeneous relation as specified below. However, to demonstrate our ideas, in this work, we consider Eq. (3) to be one of the equations in the family of special function equations, namely Hermite's equation, Legendre's equation and Laguerre's equation, besides the basic simple harmonic oscillator equation form. We will also consider an example where transformation to a nonlinear integrable equation is also made.

To find the explicit form of $P(t,x,\dot{x})$ that is consistent with Eqs. (3) and (4), we proceed in the following way. Differentiating Eq. (2) with respect to t and using the Eqs. (3) and (4) therein, we obtain the following first-order ODE equation, that is

$$\dot{x} + h(x,t)x = \dot{w}/s. \tag{5}$$

Differentiating the above equation (5) one more time with respect to t and substituting $\ddot{x} = P(t,x,\dot{x})$ in it one obtains the following equation,

$$P(t,x,\dot{x}) + (2h + xh_x)\dot{x} + (h_t + h^2)x = Q(t,w,\dot{w})/s. \tag{6}$$

Now we demand that Q satisfies the homogenous relation, that is

$$Q(t,rx,r\dot{x}) = rQ(t,x,\dot{x}). \tag{7}$$

Then Eq. (6) can be written in the following form with the help of Eqs. (2) and (5),

$$P(t,x,\dot{x}) + (2h + xh_x)\dot{x} + (h_t + h^2)x = Q(t,x,\dot{x} + hx), \tag{8}$$

or

$$\ddot{x} = Q(t,x,\dot{x} + hx) - (2h + xh_x)\dot{x} + (h_t + h^2)x. \tag{9}$$

One may observe that in the above procedure we have kept the function $h(x,t)$ as arbitrary. Then, for different choices of Q and h, one can get different nonlinear ODEs. By utilizing the known expressions, w and h, we can derive the solution for the nonlinear ODE (9). To see this clearly, we proceed as follows.

From Eqs. (2) and (5) we obtain

$$s^{-1} = \frac{\dot{x} + h(x,t)x}{\dot{w}} = \frac{x}{w}. \tag{10}$$

Rewriting Eq. (10) we find

$$\frac{d}{dt}(\ln\frac{x}{w}) + h(x,t) = 0. \tag{11}$$

The above equation can be integrated explicitly only for specific choice of $h(x,t)$. Integrating Eq. (11) with specific form of $h(x,t)$ we can obtain the solution $x(t)$ for the nonlinear equation (9). To be specific, we make a choice

$$h(x,t) = a(t)x^n + b(t) \tag{12}$$

where n is a constant and $a(t)$ and $b(t)$ are arbitrary functions of t. Then we obtain the solution of the equation (11) as

$$x(t) = \frac{w(t)e^{-\int_0^{t'} b(t')dt'}}{\left(I + n\int_0^{t'}(a(t')w(t')^n e^{-n\int_0^{t''} b(t'')dt''})dt'\right)^{\frac{1}{n}}}. \tag{13}$$

The corresponding nonlinear ODE is

$$\begin{aligned}\ddot{x} + Q(t, x, \dot{x} + a(t)x^{n+1} + b(t)x) + [(2+n)a(t)x^n + 2b(t)]\dot{x} \\ + [a(t)_t x^n + b(t)_t + a(t)^2 x^{2n} + b(t)^2 + 2a(t)b(t)x^n]x = 0.\end{aligned} \tag{14}$$

Through the above discussed procedure, we can find the solution of the nonlinear equation (1) directly without solving it. In the following, we consider the form of Q as

$$Q(t, w, \dot{w}) = p(t)\frac{\dot{w}^2}{w} + q(t)\dot{w} + r(t)w, \tag{15}$$

so that the homogeneity relation (7) is satisfied and then discuss their nonlinear counterparts in the variable $x(t)$ and investigate their significance.

3. Applications of the method

3.1. *Isochronous solution of nonlinear ODEs*

To start with we will consider a simple model, where the equation corresponding to (3) is the linear harmonic oscillator equation. Correspondingly, we make the choice $p(t) = q(t) = 0$ and $r(t) = -\omega^2$, so that Eq. (15) is of the linear harmonic oscillator (SHO) form and the corresponding nonlinear equation (14) becomes

$$\ddot{x} = -(2h(x,t) + xh(x,t)_x)\dot{x} + (h(x,t)_t + h(x,t)^2)x - \omega^2 x, \tag{16}$$

where $h(x,t)$ is an arbitrary function of x and t to be chosen appropriately. The solution of the nonlinear equation can be obtained from the equation

$$\frac{\dot{x} + h(x,t)x}{x} = \frac{\dot{w}}{w} = \cot(\omega t + \delta), \tag{17}$$

where $w = A\sin(\omega t + \delta)$ is the general solution of the SHO equation and A and δ are arbitrary constants. For the choice $h(x,t) = a(t)x^n + b(t)$, Eq. (17) becomes the well-known Bernoulli equation of the form

$$\dot{x} = -a(t)x^{n+1} - (b(t) - \cot(\omega t + \delta))x, \tag{18}$$

and the solution of the equation is given by

$$x(t) = \frac{\sin(\omega t + \delta)e^{-\int_0^{t'} b(t')dt'}}{\left(I + n\int_0^{t'}(a(t')\sin^n(\omega t' + \delta)e^{-n\int_0^{t''} b(t'')dt''})dt'\right)^{\frac{1}{n}}}, \tag{19}$$

where I is an integration constant. Equation (19) is the general solution for the equation (16) with $h(x,t) = a(t)x^n + b(t)$, so that it takes the form

$$\begin{aligned}\ddot{x} = -[(2+n)a(t)x^n + 2b(t)]\dot{x} - [a(t)_t x^n + b(t)_t + a(t)^2 x^{2n} \\ + b(t)^2 + 2a(t)b(t)x^n]x - \omega^2 x.\end{aligned} \tag{20}$$

When $n = 2m+1$, $b(t) = 0$ and $a(t) = a_0$, the solution becomes periodic with period $T = \frac{2\pi}{\omega}$ which is the same as that of the linear harmonic oscillator and it is given as

$$x(t) = \frac{\sin(\omega t + \delta)}{\left(I + \frac{(2m+1)a_0}{2^{2m}\omega}\left[\sum_{k=0}^{m}(-1)^{m+k+1}\begin{pmatrix} 2m+1 \\ k\end{pmatrix}\frac{\cos((2m+1-2k)(\omega t + \delta))}{(2m+1-2k)}\right]\right)^{\frac{1}{2m+1}}}. \tag{21}$$

The corresponding system is called isochronous system which takes the form

$$\ddot{x} + a_0((2m+3)\dot{x} + a_0x^{2m+2})x^{2m+1} + \omega^2 x = 0. \tag{22}$$

Particularly, when $m = 0$, Eq. (22) becomes the modified Emden equation

$$\ddot{x} + 3a_0x\dot{x} + a_0^2x^3 + \omega^2 x = 0, \tag{23}$$

extensively studied by Chandrasekar et al. [19, 21], which admits the amplitude independent isochronous solution

$$x(t) = \frac{\sin(\omega t + \delta)}{I + \frac{a_0}{\omega}\cos(\omega t + \delta)}. \tag{24}$$

One can also note that on fixing $m = -3/2$ in Eq. (22), we get the well known Ermakov-Pinney equation [22, 23]

$$\ddot{x} + a_0^2x^{-3} + \omega^2 x = 0. \tag{25}$$

It admits the general solution

$$x(t) = \frac{\sin(\omega t + \delta)}{I + \frac{a_0}{\omega}\cos(\omega t + \delta)}. \tag{26}$$

3.2. Special function solution of nonlinear ODEs

To illustrate the above method further, in the following, we consider three forms of Eqs. (3), namely the Hermite equation, the Legendre equation and the Laguerre equation and identify the corresponding nonlinear equations and present their associated solutions. For simplicity, we consider the form of $h(x,t) = x$ in all the three cases.

3.2.1. Hermite equation

To begin with, let us choose the Hermite's differential equation which is given by

$$\ddot{w} - 2t\dot{w} + 2nw = 0. \tag{27}$$

Comparing Eq. (27) with Eq. (3), we find

$$Q(t, w, \dot{w}) = 2t\dot{w} - 2nw. \tag{28}$$

Substituting the above Hermite form (28) into Eq. (8), we arrive at the following nonautonomous nonlinear ODE, namely

$$\ddot{x} = 2t(\dot{x} + x^2) - 3x\dot{x} - x^3 - 2nx. \tag{29}$$

To find the solution of this nonlinear ODE we recall the Rodrigues' formula for the Hermite polynomial, that is

$$w_n(t) = (-1)^n e^{t^2}\frac{d^n}{dt^n}(e^{-t^2}). \tag{30}$$

Substituting the above solution (30) into equation (11) and integrating it, we get

$$x_n(t) = w_n(t)(I_n + \int_0^t w_n(t)dt)^{-1}, \tag{31}$$

where I_n represents the integration constant. Using the following recurrence relation,

$$dw_{n+1}/dt = 2(n+1)w_n, \tag{32}$$

Eq. (31) can be explicitly integrated to yield

$$x_n(t) = 2(n+1)w_n(t)(I_n + w_{n+1})^{-1}, \tag{33}$$

where $I_n > |m_n|$. Here $|m_n|$ is the maximum root value of the equation $w_{n+1} = 0$. Equation (33) represents the solution of the nonlinear ODE (29). Figure 1 shows the behaviour of the solution for the case $n = 0, 1, 2$.

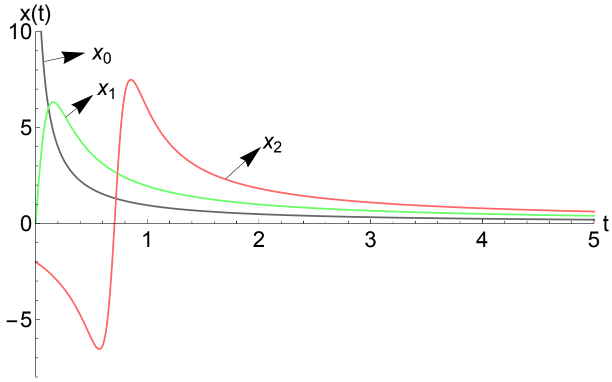

Figure 1: (Color online) The solutions of the nonlinear equation (29) for three different n values ($n = 0, 1, 2$) for $I_0 = 0.1$, $I_1 = 2.1$ and $I_2 = 6$.

3.2.2. Legendre equation

In the second choice, we choose the Legendre's differential equation

$$(1 - t^2)\ddot{w} - 2t\dot{w} + n(n+1)w = 0. \tag{34}$$

From (34), we fix the expression for Q as (vide Eq.(3))

$$Q(t, w, \dot{w}) = \frac{1}{1 - t^2}(2t\dot{w} - n(n+1))w. \tag{35}$$

Now we substitute the above Legendre form into (8). Doing so, we obtain

$$\ddot{x} = P(t, x, \dot{x}) = \frac{2t}{1 - t^2}(\dot{x} + x^2) - 3x\dot{x} - x^3 - \frac{n(n+1)}{1 - t^2}x. \tag{36}$$

To derive the solution of the above nonlinear ODE, we recall the solution of Legendre's equation in Rodrigues' form, that is

$$w_n(t) = \frac{1}{2^n n!}\frac{d^n}{dt^n}[(t^2 - 1)^n]. \tag{37}$$

Substituting the above expression into equation (11) and integrating the resultant equation, we get

$$x_n(t) = w_n(t)(I + \int_0^t w_n(t)dt)^{-1}, \tag{38}$$

where I_n represents the integration constant and w_n is the Legendre's polynomial of order n. Using the following recurrence relation

$$dw_{n+1}/dt - dw_{n-1}/dt- = (2n+1)w_n \tag{39}$$

equation (38) can be integrated explicitly to yield

$$x_n(t) = \frac{(2n+1)w_n(t)}{(I_n + w_{n+1} - w_{n-1})}, \tag{40}$$

where $I_n > |m_n|$. Here $|m_n|$ is the maximum root value of the equation $w_{n+1} - w_{n-1} = 0$. One may verify that ((40) satisfies the nonlinear ODE (36) by direct substitution. For the case $n = 0, 1, 2$ the solution (40) is plotted in Fig. 2.

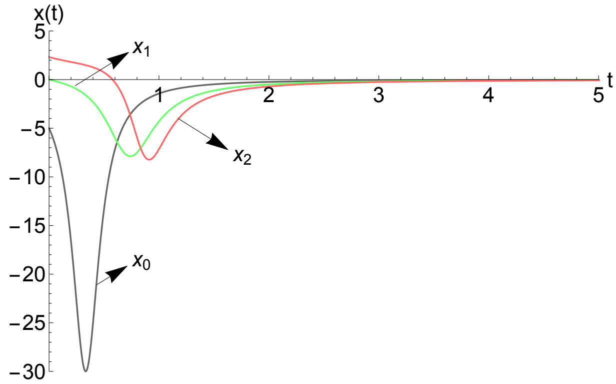

Figure 2: (Color online) The solutions of the nonlinear equation (36) for three different n values ($n = 0, 1, 2$) for $I_0 = I_1 = I_2 = -0.7$.

3.2.3. Laguerre equation

Finally we consider the Laugerre's differential equation

$$t\ddot{w} - (t-1)\dot{w} + nw = 0. \tag{41}$$

For this case the function Q reads

$$Q(t, w, \dot{w}) = \frac{1}{t}((t-1)\dot{w} - nw). \tag{42}$$

As we did in the earlier two cases, we substitute the Laguerre form (41) into equation (8). We arrive the nonlinear ODE with

$$P(t, x, \dot{x}) = \frac{(t-1)}{t}(\dot{x} + x^2) - 3x\dot{x} - x^3 - \frac{n}{t}x, \tag{43}$$

To find the solution of the nonlinear ODE in terms of the Laguerre function, we recall the Rodrigues' formula for the Laguerre function

$$w_n(t) = e^t \frac{d^n}{dt^n}(t^n e^{-t}). \tag{44}$$

Substituting this into equation (11) and integrating it we get

$$x_n(t) = w_n(t)(I + \int_0^t w_n(t)dt)^{-1}. \tag{45}$$

To integrate (45), we recall the following recurrence relation

$$(n+1)dw_n/dt - dw_{n+1}/dt- = (n+1)w_n. \tag{46}$$

Subsituting (46) into (45) and integrating , we obtain

$$x_n(t) = w_n(t)(I_n + w_n - \frac{w_{n+1}}{(n+1)})^{-1}. \tag{47}$$

where $I_n > |m_n|$ for n even and $I_n < |m_n|$ for n odd. Here $|m_n|$ is the maximum root value of the equation $w_n - \frac{w_{n+1}}{(n+1)} = 0$. The solution of the equation (43) for $n = 0, 1, 2$ is given in Fig. 3.

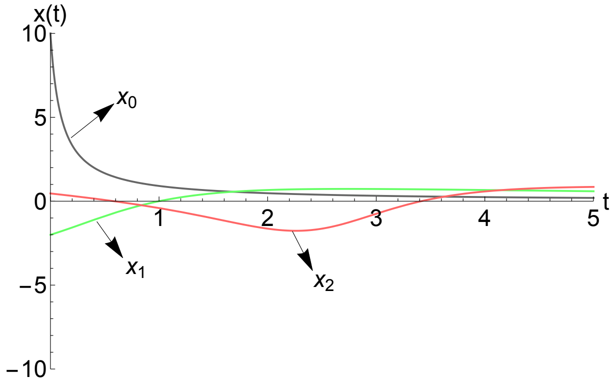

Figure 3: (Color online) The solutions of the nonlinear equation (43) for three different n values ($n = 0, 1, 2$) for $I_0 = 0.1$, $I_1 = -1$ and $I_2 = 1.5$.

3.3. Nonlinear to nonlinear ODEs

Let us consider the nonlinear equation arising in general relativity [24]

$$w\ddot{w} = 3\dot{w}^2 + \frac{w\dot{w}}{t}. \tag{48}$$

Eq. (48) admits the general solution

$$w = \sqrt{\frac{1}{(I_2 - I_1 t^2)}}, \tag{49}$$

where I_1 and I_2 are arbitrary constants. Now substituting equation (48) into (8), we get

$$x\ddot{x} = -(2h + xh_x)x\dot{x} + (h_t + h^2)x^2 + 3(\dot{x} + h + \frac{x}{t})(\dot{x} + h), \tag{50}$$

where $h = h(x, t)$, and the solution of (50) can be derived from

$$\dot{x} + h(x, t)x = \frac{\dot{w}}{w}x = -I_1 t x. \tag{51}$$

When $h(x, t) = a(t)x^n + b(t)$, Eq. (51) becomes

$$\dot{x} = -a(t)x^{n+1} - (b(t) + I_1 t)x, \tag{52}$$

and the solution of the equation (52) is given by

$$x(t) = \frac{e^{-\int_0^{t'}(b(t')+I_1 t')dt'}}{\left(I + n\int_0^{t'}(a(t')e^{-n\int_0^{t''}(b(t'')+I_1 t'')dt''})dt'\right)^{\frac{1}{n}}}. \tag{53}$$

When $a(t) = a(\text{constant})$ and $b(t) = b(\text{constant})$, the solution takes the form

$$x(t) = \frac{e^{-(bt+\frac{I_1}{2}t^2)}}{\left(I + \frac{\sqrt{n\pi}}{\sqrt{2I_1}}ae^{\frac{b^2 n}{2I_1}}\text{erf}\left[\frac{\sqrt{n}(b+I_1 t)}{\sqrt{2I_1}}\right]\right)^{\frac{1}{n}}}. \tag{54}$$

where erf is the Gauss error function. When $h(x, t) = ax^n + b$, Eq. (50) becomes

$$x\ddot{x} = -((2 + n)ax^n + 2b)x\dot{x} + (ax^n + b)^2x^2 + 3(\dot{x} + ax^n + b + +\frac{x}{t})(\dot{x} + ax^n + b). \tag{55}$$

4. Method: Third order nonlinear ODE

One can extend the above study to third order and higher order ODEs as well. Here we indicate the general method to third order ODEs only, for brevity. Let us consider a third order nonlinear ODE

$$\dddot{x} = P(t, x, \dot{x}, \ddot{x}), \tag{56}$$

where dot denotes differentiation with respect to time and P is function in t, x, $\dot{x}$and $\ddot{x}$. Let us assume that the ODE (56) admits a solution of the form

$$x(t) = w(t)/s(t), \tag{57}$$

where w is the solution of a following differential equation

$$\dddot{w} = Q(t, w, \dot{w}, \ddot{w}), \tag{58}$$

and the function $s(t)$ satisfies

$$\dot{s} = h(x, t)s. \tag{59}$$

Differentiating the equation (57) with respect to t and using the equations (58) and (59) in it we obtain the following equations, that is

$$\dot{x} + hx = \dot{w}/s. \tag{60}$$

$$\ddot{x} + (2h + xh_x)\dot{x} + (h_t + h^2)x = \ddot{w}/s. \tag{61}$$

and

$$P(t, x, \dot{x}, \ddot{x}) + (3h + xh_x)\ddot{x} + (3h_x + xh_{xx})\dot{x}^2 + (3h_t + 3h^2 + (2 + x)hh_x + (x + 1)xh_{xt})\dot{x} + (h_{tt} + 3hh_t + h^3)x = Q(t, w, \dot{w}, \ddot{w})/s. \tag{62}$$

If Q satisfies homogenous relation, that is

$$Q(t, rx, r\dot{x}, r\ddot{x}) = rQ(t, x, \dot{x}, \ddot{x}), \tag{63}$$

then equation (62) can be written as,

$$\dddot{x} = Q(t, x, \dot{x} + hx, \ddot{x} + (2h + xh_x)\dot{x} + (h_t + h^2)x) - (3h + xh_x)\ddot{x} - (3h_x + xh_{xx})\dot{x}^2 - (3h_t + 3h^2 + (2 + x)hh_x + (x + 1)xh_{xt})\dot{x} - (h_{tt} + 3hh_t + h^3)x. \tag{64}$$

The general form of Q that satisfies relation (63) is

$$Q(t, w, \dot{w}, \ddot{w}) = a(t)\frac{\dot{w}\ddot{w}}{w} + b(t)\frac{\ddot{w}^2}{w} + c(t)\frac{\dot{w}^2}{w} + d(t)\ddot{w} + e(t)\dot{w} + f(t)w. \tag{65}$$

From equations (57) and (60) we obtain

$$s^{-1} = \frac{\dot{x} + h(x, t)x}{\dot{w}} = \frac{x}{w}. \tag{66}$$

Rewriting Eq. (66), we again find

$$\frac{d}{dt}(\ln \frac{x}{w}) + h(x, t) = 0. \tag{67}$$

Using the above equations, we can again solve appropriate third order nonlinear differential equations.

5. Conclusion

In this work, we have pointed out a specific method of finding the general solution of a class nonlinear ODEs. In particular we have brought out specific forms of nonlinear ODEs that admit isochronous solutions, limit-cycle solutions and special functions as their solutions. To integrate certain integrals we recalled Rodrigues' formula and suitable recurrence relations of the considered special functions. Interestingly now one may start investigating the properties of specific families of nonlinear ODEs and their solutions. For example, it is well known that the Hermite's, Legendre's and Laguerre's equations are all Sturm-Liouville type and self adjoint equations. Now one may investigate the self adjointness and Sturm-Liouville nature of the nonlinear ODEs as well. One can extend the study to third order as well as to higher order ODEs as well. A detailed study will be presented elsewhere.

Acknowledgments

The work of V.K.C. is supported by the DST-SERB-CRG Project under Grant No. CRG/2020/004353 and V.K.C. thanks DST, New Delhi, for computational facilities under the DST-FIST program (SR/FST/PS-1/2020/135) to the Department of Physics. M.L. is supported by the DST-SERB National Science Chair program.

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