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Qualitative Analysis of Nonlinear Hilfer Fractional Implicit Differential Equations in a Banach space

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Abstract

This article focuses on the class of nonlinear implicit Hilfer-type fractional differential equations. By using the non-linear growth condition, we have derived the existence of at least one solution by applying Schauder's fixed point theorem and using Lipschitz conditions, we have derived the uniqueness of the solution with the help of the Banach contraction principle. In addition, we have discussed the stability analysis by using Ulam-Hyers and Ulam-Hyers-Rassias stabilities. All results of this paper are established in a Banach space instead of $\mathbb{R}$. We illustrate our results with the help of two examples.

Keywords: Fixed point theorems Hilfer fractional derivative implicit differential equations Ulam's stability.
2010 MSC: 34K20, 34A08, 34A09.

1. Introduction

In recent years, many researchers have paid attention to fractional calculus as it has plenteous applications in the fields of viscoelasticity, porous media, control, electromagnetic, and so on. It is more suitable for the description of many physical phenomena arising in economics, science, and engineering as compared to classical derivatives and integrals (see, for instance, [21, 24, 28, 22, 4, 20] and references therein). In

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2000, Hilfer [14] introduced the new definition of fractional derivative (known as Hilfer fractional derivative) which is a generalization of Riemann-Liouville fractional derivative as well as an interpolation between Riemann-Liouville and Caputo's fractional derivative. The authors [1, 19, 12, 30, 11] discussed the existence and uniqueness results for different types of Hilfer fractional differential equations. In [32], Yang and Wang studied the existence and uniqueness of mild solutions of Hilfer fractional evolution equations. Gu and Trujillo [13] studied the existence of a mild solution for an evolution equation with Hilfer fractional derivative. Since implicit differential equations have many applications in the various fields of science and engineering, many authors have also shown their interest in studying the different types of implicit differential equations of fractional order in [5, 2, 6, 27, 31].

On the other hand, stability theory is also an important topic in the field of mathematics. It studies the solutions of differential equations and the trajectories of dynamical systems using small perturbations. In 1940, Ulam [29] introduced Ulam's stability and studied different mathematical problems. This type of stability analysis is more suitable for a dynamical system and quite appropriate in applications where it is not possible to find an exact solution. Hyers [16] extended the results of Ulam's stability for the linear functional equation in 1941. Ulam and Hyers also investigated sufficient conditions for the stability of other types of differential equations. The stability results have been analyzed and extended by many researchers for fractional-order differential equations (see for instance [3, 17, 18, 23, 26, 7] and references therein). In [33], Zada et al. investigated Ulam-type stability for a class of fractional differential equations with non-instantaneous integral impulses. Ankit et al. [25] find the sufficient conditions for the existence, uniqueness and Ulam-Hyers stability of solutions for the coupled boundary value problem of nonlinear Caputo–Hadamard fractional differential equations associating with nonlocal integral boundary conditions. Numerous articles have been devoted to examining the stability of different FDEs; see [10, 9, 8].

In this paper, we consider the implicit differential equation involving Hilfer fractional derivative of the following form:

$$\begin{aligned} &D^{\alpha,\beta}_{0+}\varkappa(s) = A\varkappa(s) + \zeta(s, \varkappa(s), D^{\alpha,\beta}_{0+}\varkappa(s)), \quad s \in \mathcal{J} = [0, T], \\ &I^{1-\nu}_{0+}\varkappa(0) = \varkappa_0, \quad \nu = \alpha + \beta - \alpha\beta, \end{aligned} \tag{1}$$

where $D^{\alpha,\beta}_{0+}$ is the Hilfer fractional derivative of order β and type α, here $0 < \beta < 1$ and $0 \le \alpha \le 1$; A is a bounded linear operator which generates a uniformly continuous semigroup $Q(s)$ of bounded linear operators for $s \ge 0$; $\varkappa\colon \mathcal{J} \to X$ is a state function; $\zeta : \mathcal{J} \times X \times X \to X$ is the given function such that $(s)^{1-\nu}\zeta(s, x, y)$ is continuous in s, x and y; $I^{1-\nu}_{0+}$ is known as left-sided Riemann-Liouville fractional integral of order $1 - \nu$.

Motivated form [31], we derive the sufficient conditions for the existence and uniqueness results of the proposed non-linear implicit fractional differential equation (1) in a Banach space using Schauder's fixed point theorem and Banach contraction principle. Zada et al. [33] applied the Ulam stability concepts to the implicit fractional differential equations with non-instantaneous integral impulses and boundary condition. All the results of stability using the concept of the Ulam stability were established in $\mathbb{R}$ only. In this paper, we establish the existence and uniqueness results, and the stability results for nonlinear Hilfer fractional abstract implicit differential system (1) in a Banach space using the concept of the Ulam stability. To the best of our information, the Ulam stability of abstract implicit system (1) has not yet been studied in any of the research papers.

The rest of the paper is organized as follows: In section 2, we introduce some basic definitions, notations, and preliminaries results. In Section 3, we prove the existence and uniqueness results using Schauder's fixed point theorem and Banach contraction principle. In Section 4, we investigate Ulam's stability of the fractional differential equation. In the final section, the validation of our results is given through some suitable examples.

2. Preliminaries

In this section, we introduce some basic definitions, notations and preliminaries results which are to be used for our main results. Let $\mathcal{J}' = (0, T]$ and $\nu = \alpha + \beta - \alpha\beta$.

Definition 2.1 (see [24]). *The Riemann-Liouville fractional integral of order $\beta > 0$ for a function Ψ is given by*

$$I^{\beta}\Psi(s) = \frac{1}{\Gamma(\beta)} \int_0^s (s-\tau)^{\beta-1}\Psi(\tau)d\tau, \quad s > 0,$$

where Γ is the gamma function, and $\Psi \in L^1(\mathcal{J}, X)$.

Definition 2.2 (see [24]). *The Riemann-Liouville fractional derivative of order $0 \le n-1 < \beta < n$ for a function Ψ is defined as*

$$D^{\beta}\Psi(s) = \frac{1}{\Gamma(n-q)} \frac{d^n}{ds^n} \int_0^s \frac{\Psi(\tau)}{(s-\tau)^{q+1-n}} d\tau, \quad s > 0.$$

Definition 2.3 ([14]). *The Hilfer fractional derivative of type $0 \le \alpha \le 1$ and of order $0 < \beta < 1$ with lower limit 0 for a function Ψ is defined to be*

$$D_{0+}^{\alpha,\beta}\Psi(s) = \left(I_{0+}^{\alpha(1-\beta)} \frac{d}{ds}(I_{0+}^{(1-\nu)}\Psi)\right)(s) = \left(I_{0+}^{\alpha(1-\beta)}(D^{\nu}\Psi)\right)(s),$$

provided that the right hand side exists.

Remark 2.4. *If $\alpha = 0$ and $0 < \beta < 1$, then Hilfer fractional derivative becomes Riemann-Liouville fractional derivative of order β.*

Throughout this paper we assume that $C(\mathcal{J}, X)$ is a Banach Space of all continuous functions from $\mathcal{J}$ into X with the sup-norm $\|\cdot\|_C$ and $L^1(\mathcal{J}, X)$ is a Banach space of Lebesgue-integral functions from $\mathcal{J}$ into X endowed with the norm

$$\|\Psi\|_1 = \int_0^T \|\Psi(\tau)\|\, d\tau.$$

Define $\mathcal{Y}_\nu = \mathcal{C}_\nu(\mathcal{J}, X) = \{\varkappa\colon \mathcal{J} \to X \mid \varkappa\colon \mathcal{J}' \to X$ is continuous and $\lim_{s\to 0} s^{1-\nu}\varkappa(s)$ exists and finite$\}$. Clearly $\mathcal{Y}_\nu$ is a Banach space endowed with norm $\|\varkappa\|_{\mathcal{Y}_\nu} = \sup_{s\in\mathcal{J}} \|s^{1-\nu}\varkappa(s)\|$, $\varkappa \in \mathcal{Y}_\nu$.

Let's define Banach spaces $\mathcal{Y}_\nu^{\beta,\nu}$ and $\mathcal{Y}_\nu^{\nu}$ as below:

$$\mathcal{Y}_\nu^{\beta,\nu} = \{\varkappa \in \mathcal{Y}_\nu, D_{0+}^{\beta,\nu}\varkappa \in \mathcal{Y}_\nu\}$$

and

$$\mathcal{Y}_\nu^{\nu} = \{\varkappa \in \mathcal{Y}_\nu, D_{0+}^{\nu}\varkappa \in \mathcal{Y}_\nu\}.$$

Moreover, $\mathcal{Y}_\nu^0 = \mathcal{Y}_\nu$ and it is easy to see that

$$\mathcal{Y}_\nu^{\nu} \subset \mathcal{Y}_\nu^{\beta,\nu}.$$

If we suppose that $\zeta(\cdot, \varkappa(\cdot)) \in C_\nu(J, X)$ for each $\varkappa \in C_\nu(J, X)$, then we say from the paper [34] that a function $\varkappa \in \mathcal{Y}_\nu^{\nu}$ is a solution of fractional initial value problem:

$$\begin{cases} D_{0+}^{\alpha,\beta}\varkappa(s) = A\varkappa(s) + \zeta(s, \varkappa(s)), 0 < \beta < 1,\ 0 \le \alpha \le 1 \\ I_{0+}^{1-\nu}\varkappa(0) = \varkappa_0, \quad \nu = \alpha + \beta - \alpha\beta, \end{cases}$$

if and only if $\varkappa(\cdot)$ satisfies the following:

$$\varkappa(t) = \frac{s^{\nu-1}}{\nu}\varkappa_0 + \frac{1}{\beta}\int_0^s (s-\tau)^{\beta-1}[A\varkappa(r) + \zeta(r, \varkappa(r))]dr. \tag{2}$$

Lemma 2.5 ([13]). *The equation* (2) *is equivalent to the following*

$$\varkappa(s) = U_{\alpha,\beta}(s)\varkappa_0 + \int_0^s R_\beta(s-\tau)\zeta(\tau,\varkappa(\tau))d\tau, \quad s \in \mathcal{J}' \tag{3}$$

where

$$U_{\alpha,\beta}(s) = I_{0+}^{\alpha(1-\beta)}R_\beta(s), \qquad R_\beta(s) = (s)^{\beta-1}P_\beta(s),$$
$$P_\beta(s) = \beta\int_0^\infty \theta M_\beta(\theta)Q(s^\beta\theta)d\theta, \qquad M_\beta(\theta) = \frac{1}{\varrho}\theta^{-1-1/\varrho}\xi_\varrho(\theta^{-1/\varrho})$$
$$\xi_\varrho(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-n\varrho-1}\frac{\Gamma(n\varrho+1)}{n!}\sin(n\pi\varrho), \qquad \theta \in (0,\infty).$$

At end of this section, we state the generalized Gronwall's lemma for singular kernels that will be used in the following sections.

Lemma 2.6 ([15]). *Let $a > 0$ and $0 < \alpha < 1$ be a constants and let $\vartheta : [0,T] \to [0,\infty)$ be a real function and $\xi(\cdot)$ is a non-negative and locally integrable function on $[0,T]$ such that*

$$\vartheta(s) \le \xi(s) + a\int_0^s (s-\tau)^{-\alpha}\vartheta(\tau)\,d\tau.$$

Then, there is a constant $\kappa = \kappa(\alpha)$ such that

$$\vartheta(s) \le \xi(s) + \kappa a\int_0^s (s-\tau)^{-\alpha}\xi(\tau)\,d\tau,$$

for every $s \in [0,T]$.

3. Existence and Uniqueness Analysis

In this section, we establish the existence and uniqueness results of the proposed non-linear implicit fractional differential equation (1) using Schauder's fixed point theorem and Banach contraction principle.

We assume throughout this paper that there exists $M \ge 1$ such that $\|Q(s)\| \le M$ for each $s \ge 0$. Then we derive from [13, Proposition 2.16 and 2.17] that $R_\beta(s)_{s>0}$ and $U_{\alpha,\beta}(s)_{s>0}$ are strongly continuous linear operators, and for any $z \in X$ and $s > 0$ we have

$$\|R_\beta(s)z\| \le \frac{Ms^{\beta-1}}{\Gamma(\beta)}\|z\| \quad \text{and} \quad \|U_{\alpha,\beta}(s)z\| \le \frac{Ms^{(\alpha-1)(\beta-1)}}{\Gamma(\alpha(1-\beta)+\beta)}\|z\|.$$

For our convenience, we write $K_\varkappa(s) = \zeta\left(s,\varkappa(s),D_{0+}^{\alpha,\beta}\varkappa(s)\right)$. We now define the set E_k and E_k^ν as

$$E_k = \{y(\cdot) \in C(\mathcal{J},X) \colon \|y\|_C \le k\} \quad \text{and} \quad E_k^\nu = \{\varkappa \in \mathcal{Y}_\nu \colon \|\varkappa\|_{\mathcal{Y}_\nu} \le k\}.$$

We now make the following hypothesis for the existence of mild solution of (1).

(H1) There are functions $p_1, p_2, p_3 \in \mathcal{C}_\nu(\mathcal{J},\mathbb{R}_+)$ with $p_1^* = \sup_{s\in\mathcal{J}} p_1(s) < \infty$, $p_2^* = \sup_{s\in\mathcal{J}} p_2(s) < 1$ and $p_3^* = \sup_{s\in\mathcal{J}} p_3(s) < 1$ such that

$$\|\zeta(s,x,y)\| \le p_1(s) + p_2(s)\|x\| + p_3(s)\|y\|$$

for $s \in \mathcal{J}$ and $x, y \in X$.

Theorem 3.1. *If the hypothesis (H1) is satisfied and*

$$\frac{MT^{\beta}\mathcal{B}(\beta,\nu)}{\Gamma(\beta)}\left[\frac{p_3^*\|A\|+p_2^*}{1-p_3^*}\right]<1, \tag{4}$$

here $\mathcal{B}(\cdot,\cdot)$ is a beta function, then the system (1) has a mild solution in $\mathcal{Y}_{\nu}^{\nu}\subset\mathcal{Y}_{\nu}^{\alpha,\beta}$.

Proof. Consider an operator $P\colon\mathcal{Y}_{\nu}\to\mathcal{Y}_{\nu}$ defined by

$$(P\varkappa)(s)=U_{\alpha,\beta}(s)\varkappa_0+\int_0^s R_{\beta}(s-\tau)\zeta\left(\tau,\varkappa(\tau),D_{0+}^{\alpha,\beta}\varkappa(\tau)\right)d\tau,\quad s\in(0,T]. \tag{5}$$

For any $y\in C(\mathcal{J},X)$, we set $\varkappa(s)=s^{\nu-1}y(s)$, $s\in\mathcal{J}'$. we define a map $\digamma$ as

$$(\digamma y)(s)=\digamma(s^{1-v}\varkappa)(s)=\begin{cases}s^{1-\nu}(P\varkappa)(s), & s\in\mathcal{J}',\\ \frac{\varkappa_0}{\Gamma(\alpha(1-\beta)+\beta)}, & s=0,\end{cases} \tag{6}$$

It is obvious that y is any fixed point of map $\digamma$ if and only if $\varkappa$ is a fixed point of P, which is further equivalent to say that $\varkappa$ is the mild solution of (1).

The proof is divided into following steps:

Step I. $\digamma$ is map from E_k to E_k for some $k>0$.

On contrary, we suppose that this is not true for each $k>0$. Thus there exists $y_k\in E_k$ for each $k>0$ such that $\|\digamma(y_k)(s_k)\|>k$ for some $s_k\in\mathcal{J}$. Let $\varkappa_k(s)=t^{\nu-1}y_k(s)$, $s\in\mathcal{J}'$.

$$k<\|\digamma(y_k)(s_k)\|\le\frac{M\varkappa_0}{\Gamma(\alpha(1-\beta)+\beta)}+\frac{M{s_k}^{1-\nu}}{\Gamma(\beta)}\int_0^{s_k}(s_k-\tau)^{\beta-1}\|\zeta\left(\tau,\varkappa(\tau),D_{0+}^{\alpha,\beta}\varkappa(\tau)\right)\|\,d\tau \tag{7}$$

But for any $s\in\mathcal{J}$ we have

$$\begin{aligned}\|D_{0+}^{\alpha,\beta}\varkappa(s)\|-\|A\varkappa(s)\|\le\|D_{0+}^{\alpha,\beta}\varkappa(s)-A\varkappa(s)\|&=\|\zeta\left(s,\varkappa(s),D_{0+}^{\alpha,\beta}\varkappa(s)\right)\|\\ &\le p_1(s)+p_2(s)\|\varkappa(s)\|+p_3(s)\|D_{0+}^{\alpha,\beta}\varkappa(s)\|.\end{aligned}$$

Thus we have

$$\begin{aligned}\|D_{0+}^{\alpha,\beta}\varkappa(s)\|&\le\|A\varkappa(s)\|+p_1(s)+p_2(s)\|\varkappa(s)\|+p_3(s)\|D_{0+}^{\alpha,\beta}\varkappa(s)\|\\ &\le\frac{p_1^*+[p_3^*\|A\|+p_2^*]\,\|\varkappa(s)\|}{1-p_3^*}.\end{aligned} \tag{8}$$

Therefore we obtain that

$$\begin{aligned}\|K_{\varkappa}(s)\|=\|\zeta\left(s,\varkappa(s),D_{0+}^{\alpha,\beta}\varkappa(s))\right)\|&\le p_1(s)+p_2(s)\|\varkappa(s)\|+p_3(s)\|D_{0+}^{\alpha,\beta}\varkappa(s)\|\\ &\le\frac{p_1^*+[p_3^*\|A\|+p_2^*]\,\|\varkappa(s)\|}{1-p_3^*}.\end{aligned} \tag{9}$$

We now obtain from (7) and (9) that

$$k<\|\digamma(y_k)(s_k)\|\le\frac{M\varkappa_0}{\Gamma(\alpha(1-\beta)+\beta)}+\frac{Ms_k^{1-\alpha+\alpha\beta}}{\Gamma(\beta+1)}\frac{p_1^*}{1-p_3^*}+\frac{Ms_k^{\beta}\mathcal{B}(\beta,\nu)}{\Gamma(\beta)}\left[\frac{p_3^*\|A\|+p_2^*}{1-p_3^*}\right]\|\varkappa\|_{\mathcal{Y}_{\nu}}, \tag{10}$$

Dividing both sides of (10) by k and then taking $k\to\infty$, we obtain

$$1<\frac{MT^{\beta}\mathcal{B}(\beta,\nu)}{\Gamma(\beta)}\left[\frac{p_3^*\|A\|+p_2^*}{1-p_3^*}\right].$$

This gives us a contradiction. Thus there is a $k>0$ such that $\digamma(E_k)\subset E_k$.

Step II. $\digamma$ is Continuous.

Let $\{y_n\}$ be a sequence in $C(\mathcal{J},X)$ such that $y_n\to y\in C(\mathcal{J},X)$. Take $\varkappa_n(s)=s^{\nu-1}y_n(s)$ and $\varkappa(s)=s^{\nu-1}y(s)$, $s\in\mathcal{J}'$. Then for each $s\in\mathcal{J}$ we have

$$\begin{aligned}\|\digamma(y_n)(s)-\digamma(y)(s)\| &\le \left\|\int_0^s s^{1-\nu}R_\beta(s-\tau)\left[\zeta\left(\tau,\varkappa_n(\tau),D_{0+}^{\alpha,\beta}\varkappa_n(\tau)\right)-\zeta\left(\tau,\varkappa(\tau),D_{0+}^{\alpha,\beta}\varkappa(\tau)\right)\right]d\tau\right\|\\ &\le \frac{Ms^{1-\nu}}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}\tau^{\nu-1}\left\|\tau^{1-\nu}[K_{\varkappa_n}(\tau)-K_{\varkappa}(\tau)]\right\|d\tau\\ &\le \frac{Ms^{1-\nu}}{\Gamma(\beta)}\left(\int_0^s(s-\tau)^{\beta-1}\tau^{\nu-1}d\tau\right)\|K_{\varkappa_n}-K_{\varkappa}\|_{\mathcal{Y}_\nu}\\ &= \frac{MT^\beta}{\Gamma(\beta)}\mathcal{B}(\beta,\nu)\|K_{\varkappa_n}-K_{\varkappa}\|_{\mathcal{Y}_\nu},\end{aligned}\tag{11}$$

where $\mathcal{B}(\beta,\nu)$ is a beta function. Since $(s)^{1-\nu}\zeta(s,x,y)$ is continuous in s, x and y, we obtain that $\|\digamma(y_n)(s)-\digamma(y)(s)\|\to 0$ as $n\to\infty$.

Step III. In next step, we prove that $\digamma(E_k)$ is equicontinuous.

Let $0\le s_1<s_2\le T$ and $\varkappa\in E_k$, we have

$$\begin{aligned}\|\digamma(y)(s_2)-\digamma(y)(s_1)\| \le\ & \left\|s_2^{1-\nu}U_{\alpha,\beta}(s_2)\varkappa_0-s_1^{1-\nu}U_{\alpha,\beta}(s_1)\varkappa_0\right\|\\ &+\left\|\int_0^{s_1}\left[s_2^{1-\nu}R_\beta(s_2-\tau)-s_1^{1-\nu}R_\beta(s_1-\tau)\right]\zeta\left(\tau,\varkappa(\tau),D_{0+}^{\alpha,\beta}\varkappa(\tau)\right)d\tau\right\|\\ &+\frac{Ms_2^{1-\nu}}{\Gamma(\beta)}\left[\frac{p_1^*}{1-p_3^*}+\frac{(p_3^*\|A\|+p_2^*)s_1^{\nu-1}}{1-p_3^*}\|\varkappa\|_{\mathcal{Y}_\nu}\right]\int_{s_1}^{s_2}(s_2-\tau)^{\beta-1}d\tau\\ =\ & I_1+I_2+I_3,\end{aligned}$$

where

$$\begin{aligned}I_1 &= \left\|s_2^{1-\nu}U_{\alpha,\beta}(s_2)\varkappa_0-s_1^{1-\nu}U_{\alpha,\beta}(s_1)\varkappa_0\right\|,\\ I_2 &= \left\|\int_0^{s_1}\left[s_2^{1-\nu}R_\beta(s_2-\tau)-s_1^{1-\nu}R_\beta(s_1-\tau)\right]\zeta\left(\tau,\varkappa(\tau),D_{0+}^{\alpha,\beta}\varkappa(\tau)\right)d\tau\right\|\\ I_3 &= \frac{Ms_2^{1-\nu}}{\Gamma(\beta)}\left[\frac{p_1^*}{1-p_3^*}+\frac{(p_3^*\|A\|+p_2^*)s_1^{\nu-1}}{1-p_3^*}\|\varkappa\|_{\mathcal{Y}_\nu}\right]\int_{s_1}^{s_2}(s_2-\tau)^{\beta-1}d\tau.\end{aligned}$$

From expression of I_3, we can easily see that I_3 tend to 0 as $s_1\to s_2$ independent of $\varkappa\in E_k^\nu$, that is, independent of $y\in E_k$.

$$\begin{aligned}I_1 =\ & \frac{1}{\Gamma(\alpha(1-\beta))}\left\|s_2^{1-\nu}\int_0^{s_2}(s_2-\tau)^{\alpha(1-\beta)-1}\tau^{\beta-1}P_\beta(\tau)\varkappa_0\,d\tau\right.\\ &\left.-s_1^{1-\nu}\int_0^{s_1}(s_1-\tau)^{\alpha(1-\beta)-1}\tau^{\beta-1}P_\beta(\tau)\varkappa_0\,d\tau\right\|\\ \le\ & \frac{1}{\Gamma(\alpha(1-\beta))}\int_0^{s_1}\left|s_2^{1-\nu}(s_2-\tau)^{\alpha(1-\beta)-1}-s_1^{1-\nu}(s_1-\tau)^{\alpha(1-\beta)-1}\right|\tau^{\beta-1}\|P_\beta(\tau)\varkappa_0\|\,d\tau\\ &+\frac{s_2^{1-\nu}}{\Gamma(\alpha(1-\beta))}\int_{s_1}^{s_2}(s_2-\tau)^{\alpha(1-\beta)-1}\tau^{\beta-1}\|P_\beta(\tau)\varkappa_0\|\,d\tau\\ \le\ & \frac{M\|\varkappa_0\|}{\Gamma(\alpha(1-\beta))\Gamma(\beta)}\int_0^{s_1}\left|\left[s_2^{1-\nu}(s_2-\tau)^{\alpha(1-\beta)-1}-s_1^{1-\nu}(s_1-\tau)^{\alpha(1-\beta)-1}\right]\tau^{\beta-1}\right|d\tau\\ &+\frac{M\|\varkappa_0\|s_1^{\beta-1}s_2^{1-\nu}}{\Gamma(\alpha(1-\beta))\Gamma(\beta)}\frac{(s_2-s_1)^{\alpha(1-\beta)}}{\alpha(1-\beta)}\\ &\to 0 \text{ as } s_2\to s_1 \text{ independent of } \varkappa.\end{aligned}$$

We also obtain from (9) that

$$
\begin{aligned}
I_2 \leq & \left\| \int_0^{s_1} \left[s_2^{1-\nu}(s_2-\tau)^{\beta-1} - s_1^{1-\nu}(s_1-\tau)^{\beta-1} \right] P_\beta(s_2-\tau) K_\varkappa(\tau) d\tau \right\| \\
& + \left\| \int_0^{s_1} s_1^{1-\nu}(s_1-\tau)^{\beta-1} \left[P_\beta(s_2-\tau) - P_\beta(s_1-\tau) \right] K_\varkappa(\tau) d\tau \right\| \\
& + \left\| \int_{s_1}^{s_2} s_2^{1-\nu}(s_2-\tau)^{\beta-1} \left[P_\beta(s_2-\tau) K_\varkappa(\tau) \right] d\tau \right\| \\
\leq & \frac{p_1^* + (p_3^*\|A\| + p_2^*)\|\varkappa\|_{\mathcal{Y}_\nu}}{1-p_3^*} \left[\frac{M}{\gamma(\beta)} \int_0^{s_1} \left| s_2^{1-\nu}(s_2-\tau)^{\beta-1} - s_1^{1-\nu}(s_1-\tau)^{\beta-1} \right| \tau^{\nu-1} d\tau \right. \\
& \left. + \int_0^{s_1} s_1^{1-\nu}(s_1-\tau)^{\beta-1}\tau^{\nu-1} \left\| P_\beta(s_2-\tau) - P_\beta(s_1-\tau) \right\| d\tau + \frac{M s_1^{\nu-1}}{\gamma(\beta)} \int_{s_1}^{s_2} s_2^{1-\nu}(s_2-\tau)^{\beta-1} d\tau \right] \\
& \to 0 \text{ as } s_2 \to s_1 \text{ independent of } \varkappa.
\end{aligned}
$$

Thus $\digamma(E_k)$ is equicontinous on $[0,T]$. Hence we can say from Arzela-Ascoli theorem that the map $\digamma$ is completely continuous on E_k. Finally Schauder's fixed point theorem gives that $\digamma$ has a fixed point on E_k. Hence we say that the system (1) has a mild solution on E_k^ν. □

We now make the following hypothesis for the uniqueness of mild solution of (1)

(H2) The map $\zeta(\cdot, x, y)\colon \mathcal{J} \times X \times X \to X$ is measurable and there exist positive constant $L_1 > 0$ and $L_2 > 0$ such that

$$\|\zeta(s,x,y) - \zeta(s,\bar{x},\bar{y})\| \leq L_1\|x-\bar{x}\| + L_2\|y-\bar{y}\|$$

for any $s \in \mathcal{J}$ and $x, y, \bar{x}, \bar{y} \in X$.

Theorem 3.2. *If (H2) and*

$$\frac{MT^\beta(L_1 + \|A\|L_2)}{(1-L_2)\Gamma(\beta)} \mathcal{B}(\beta,\nu) < 1, \tag{12}$$

hold, then the system (1) *has a unique mild solution.*

Proof. We consider the map P and $\digamma$ that are defined by (5) and (6) respectively. We shall now show that the map $\digamma$ is a contraction on $C(\mathcal{J}, X)$. For any $y_1, y_2 \in C(\mathcal{J}, X)$, we set $\varkappa_i(s) = s^{\nu-1}y_i(s)$, $s \in \mathcal{J}'$, i=1,2. We get

$$
\begin{aligned}
\|D_{0+}^{\alpha,\beta}\varkappa_2(s) - D_{0+}^{\alpha,\beta}\varkappa_1(s)\| &\leq \|A[\varkappa_2 - \varkappa_1]\| + \left\| \zeta\left(s, \varkappa_2(s), D_{0+}^{\alpha,\beta}\varkappa_2(s)\right) - \zeta\left(s, \varkappa_1(s), D_{0+}^{\alpha,\beta}\varkappa_1(s)\right) \right\| \\
&\leq \frac{\|A\| + L_1}{1-L_2} \|\varkappa_2(s) - \varkappa_1(s)\|.
\end{aligned}
$$

Thus we obtain

$$
\begin{aligned}
\|K_{\varkappa_2}(s) - K_{\varkappa_1}(s)\| &\leq L_1\|\varkappa_2 - \varkappa_1\| + L_2\|D_{0+}^{\alpha,\beta}\varkappa_2(s) - D_{0+}^{\alpha,\beta}\varkappa_1(s)\| \\
&\leq \frac{L_1 + \|A\|L_2}{1-L_2} \|\varkappa_2(s) - \varkappa_1(s)\|.
\end{aligned} \tag{13}
$$

Now we have

$$\|\digamma(y_2)(s)-\digamma(y_1)(s)\| \le \left\| \int_0^s s^{1-\nu} R_\beta(s-\tau)\left[\zeta\left(\tau,\varkappa_2(\tau),D_{0+}^{\alpha,\beta}\varkappa_2(\tau)\right)\right.\right. \tag{14}$$

$$\left.\left. -\zeta\left(\tau,\varkappa_1(\tau),D_{0+}^{\alpha,\beta}\varkappa_1(\tau)\right)\right] d\tau \right\|$$

$$\le \frac{Ms^{1-\nu}(L_1+\|A\|L_2)}{(1-L_2)\Gamma(\beta)}\int_0^s (s-\tau)^{1-\beta}\|\varkappa_2(\tau)-\varkappa_1(\tau)\|\, d\tau$$

$$\le \frac{Ms^{1-\nu}(L_1+\|A\|L_2)}{(1-L_2)\Gamma(\beta)}\int_0^s (s-\tau)^{1-\beta}\tau^{\nu-1}\|\tau^{1-\nu}\varkappa_2(\tau)-\tau^{1-\nu}\varkappa_1(\tau)\|\, d\tau$$

$$\le \frac{MT^{\beta}(L_1+\|A\|L_2)}{(1-L_2)\Gamma(\beta)}\mathcal{B}(\beta,\nu)\|y_2-y_1\|_C. \tag{15}$$

Thus we get

$$\|\digamma(y_2)-\digamma(y_1)\|_C \le \frac{MT^{\beta}(L_1+\|A\|L_2)}{(1-L_2)\Gamma(\beta)}\mathcal{B}(\beta,\nu)\|y_2-y_1\|_C$$

The map $\digamma$ is contraction on E_k. Banach contraction principle ensures that the map $\digamma$ has a fixed point on E_k, that is, the system (1) has a unique mild solution on E_k^{ν}. □

4. Stability Analysis

In this section, we shall derive the various type of Ulam's stability for system (1).

We firstly consider the following implicit differential equation involving Hilfer fractional derivative:

$$D_{0+}^{\alpha,\beta}\varkappa(s) = A\varkappa(s)+\zeta(s,\varkappa(s),D_{0+}^{\alpha,\beta}\varkappa(s)), \quad s\in\mathcal{J}. \tag{16}$$

Let $\varepsilon>0$ and $\psi\in C(\mathcal{J},[0,\infty))$. We will use the Ulam stability concepts for the system (16) that was used by Zada et al. [33]. We now consider the following inequalities:

$$\|D_{0+}^{\alpha,\beta}\xi(s)-A\xi(s)-\zeta(s,\xi(s),D_{0+}^{\alpha,\beta}\xi(s))\| \le \varepsilon, \quad s\in\mathcal{J}, \tag{17}$$

$$\|D_{0+}^{\alpha,\beta}\xi(s)-A\xi(s)-\zeta(s,\xi(s),D_{0+}^{\alpha,\beta}\xi(s))\| \le \varepsilon\psi(s), \quad s\in\mathcal{J}, \tag{18}$$

$$\|D_{0+}^{\alpha,\beta}\xi(s)-A\xi(s)-\zeta(s,\xi(s),D_{0+}^{\alpha,\beta}\xi(s))\| \le \psi(s), \quad s\in\mathcal{J}. \tag{19}$$

Definition 4.1. *The system (1) is said to be Ulam-Hyers stable if there is a constant $G_f>0$ such that for each $\varepsilon>0$ and for each solution $\xi\in\mathcal{Y}_\nu^\nu$ of the inequality (17) there is a solution $\varkappa\in\mathcal{Y}_\nu^\nu$ of equation (1) satisfying*

$$\|\xi(s)-\varkappa(s)\| \le G_f\varepsilon, \quad s\in\mathcal{J}.$$

Definition 4.2. *The system (1) is said to be generalized Ulam-Hyers stable if there is a function $\phi_f\in C([0,\infty),[0,\infty)), \phi_f(0)=0$ such that for each solution $\xi\in\mathcal{Y}_\nu^\nu$ of the inequality (17), there is a solution $\varkappa\in\mathcal{Y}_\nu^\nu$ of equation (1) satisfying*

$$\|\xi(s)-\varkappa(s)\| \le \phi_f(\varepsilon), \quad s\in\mathcal{J}.$$

Definition 4.3. *The system (1) is said to be Ulam-Hyers-Rassias stable with respect to $\psi\in\mathcal{C}_\nu(\mathcal{J},R)$ if there is a constant $G_{f,\psi}>0$ such that for each $\varepsilon>0$ and for each solution $\xi\in\mathcal{Y}_\nu^\nu$ of the inequality (19), there is a solution $\varkappa\in\mathcal{Y}_\nu^\nu$ of equation (1) satisfying*

$$\|\xi(s)-\varkappa(s)\| \le G_{f,\psi}\psi(s)\varepsilon, \quad s\in\mathcal{J}.$$

Definition 4.4. *The system* (1) *is said to be generalized Ulam-Hyers-Rassias stable with respect to* $\psi \in \mathcal{C}_\nu(\mathcal{J}, R)$ *if there is a constant* $G_{f,\psi} > 0$ *such that for each solution* $\xi \in \mathcal{Y}_\nu^\nu$ *of the inequality* (18)*, there is a solution* $\varkappa \in \mathcal{Y}_\nu^\nu$ *of equation* (1) *satisfying*

$$\|\xi(s) - \varkappa(s)\| \leq G_{f,\psi}\psi_f(s), \quad s \in \mathcal{J}.$$

Remark 4.5. *Any function* $\xi \in \mathcal{Y}_\nu^\nu$ *satisfies the inequality*

$$\|D_{0+}^{\alpha,\beta}\xi(s) - A\xi(s) - \zeta(s, \xi(s), D_{0+}^{\alpha,\beta}\xi(s))\| \leq \varepsilon, \quad s \in \mathcal{J},$$

if and only if there is a function $g \in \mathcal{Y}_\nu^\nu$ *such that*
(i) $\|g(s)\| \leq \varepsilon, s \in \mathcal{J}$,
(ii) $D_{0+}^{\alpha,\beta}\xi(s) = A\xi(s) + \zeta(s, \xi(s), D_{0+}^{\alpha,\beta}\xi(s)) + g(s), \quad s \in \mathcal{J}$.

We can easily see that

1. If $\psi(s) = 1$, then Definition 4.3 is same as Definition 4.1.
2. Definition 4.1 $\Rightarrow$ Definition 4.2
3. Definition 4.3 $\Rightarrow$ Definition 4.4.

Lemma 4.6. *Let* $\xi \in \mathcal{Y}_\nu^\nu$ *be a solution of* (17)*. Then* ξ *satisfies the following integral inequality:*

$$\left\|\xi(s) - U_{\alpha,\beta}(s)\xi_0 - \int_0^s R_\beta(s-\tau)K_\xi(\tau)d\tau\right\| \leq \frac{MT^\beta}{\Gamma(\beta+1)}\varepsilon.$$

Proof. Since $\xi \in \mathcal{Y}_\nu^\nu$ is a solution of (17), we obtain from Remarks (4.5) that

$$D_{0+}^{\alpha,\beta}\xi(s)) = A\xi(s) + K_\xi(s) + g(s).$$

Therefore we get

$$\xi(s) = U_{\alpha,\beta}(s)\xi(0) + \int_0^s R_\beta(s-\tau)[K_\xi(\tau) + g(\tau)]d\tau$$

and

$$\begin{aligned}\left\|\xi(s) - U_{\alpha,\beta}(s)\xi(0) - \int_0^s R_\beta(s-\tau)K_\xi(\tau)d\tau\right\| &= \left\|\int_0^s R_\beta(s-\tau)g(\tau)d\tau\right\| \\ &\leq \frac{M}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}\|g(\tau)\|d\tau \\ &\leq \frac{Ms^\beta}{\Gamma(\beta+1)}\varepsilon, \quad s \in \mathcal{J}.\end{aligned}$$

Hence

$$\left\|\xi(s) - U_{\alpha,\beta}(s)\xi(0) - \int_0^s R_\beta(s-\tau)K_\xi(\tau)d\tau\right\| \leq \frac{MT^\beta}{\Gamma(\beta+1)}\varepsilon.$$

□

Now we are going to prove the stability result for the system (1).

Theorem 4.7. *If all hypothesis of Theorem 3.2 are satisfied, then the system* (1) *is Ulam-Hyers stable.*

Proof. Take $\varepsilon > 0$. We now suppose that the function $\xi \in \mathcal{Y}_\nu^\nu$ satisfies the following inequality

$$\|D_{0+}^{\alpha,\beta}\xi(s) - A\xi(s) - \zeta(s, \xi(s), D_{0+}^{\alpha,\beta}\xi(s))\| \leq \varepsilon, \quad s \in \mathcal{J} \tag{20}$$

and $I_{0+}^{1-\nu}\xi(0) = \xi_0$. Since all the conditions of Theorem 3.2 are satisfied, the system (1) has a unique mild solution. If we let $\varkappa \in C_{1-v}^v[\mathcal{J}, X]$ be the unique solution of implicit system (1), then we have

$$\varkappa(s) = U_{\alpha,\beta}(s)\varkappa_0 + \int_0^s R_\beta(s-\tau)\zeta(\tau, \varkappa(\tau), D_{0+}^{\alpha,\beta}\varkappa(\tau))d\tau.$$

Clearly, if $\xi_0 = \varkappa_0$, then $I_{0+}^{1-v}\xi(0) = I_{0+}^{1-v}\varkappa(0)$ and

$$U_{\alpha,\beta}(s)\xi_0 = U_{\alpha,\beta}(s)\varkappa_0.$$

We also obtain from Lemma 4.6 that

$$\left\|\xi(s) - U_{\alpha,\beta}(s)\xi_0 - \int_0^s R_\beta(s-\tau)K_\xi(\tau)d\tau\right\| \leq \frac{MT^\beta}{\Gamma(\beta+1)}\varepsilon.$$

For any $s \in \mathcal{J}$, we have

$$\begin{aligned}
\|\xi(s) - \varkappa(s)\| &= \left\|\xi(s) - U_{\alpha,\beta}(s)\varkappa_0 - \int_0^s R_\beta(s-\tau)K_\varkappa(\tau)d\tau\right\| \\
&\leq \left\|\xi(s) - U_{\alpha,\beta}(s)\xi_0 - \int_0^s R_\beta(s-\tau)K_\xi(\tau)d\tau\right\| \\
&+ \left\|\int_0^s R_\beta(s-\tau)K_\xi(\tau)d\tau - \int_0^s R_\beta(s-\tau)K_\varkappa(\tau)d\tau\right\| \\
&\leq \frac{MT^\beta}{\Gamma(\beta+1)}\varepsilon + \frac{M}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}\frac{L_1 + \|A\|L_2}{1-L_2}\|\xi(\tau) - \varkappa(\tau)\|d\tau
\end{aligned}$$

By Lemma 2.6, we get

$$\begin{aligned}
\|\xi(s) - \varkappa(s)\| &\leq \frac{MT^\beta}{\Gamma(\beta+1)}\varepsilon + \frac{Mh}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}\frac{L_1 + \|A\|L_2}{1-L_2}\frac{MT^\beta}{\Gamma(\beta+1)}\varepsilon d\tau \\
&\leq \left[1 + \frac{MhT^\beta(L_1 + \|A\|L_2)}{(1-L_2)\Gamma(\beta+1)}\right]\frac{MT^\beta}{\Gamma(\beta+1)}\varepsilon \\
&= G_f\varepsilon,
\end{aligned}$$

where $h = h(\beta)$ is some constant. Hence, the system (1) is Ulam-Hyers stable. In addition, if we take $\phi(\varepsilon) = G_f\varepsilon$, then $\phi(0) = 0$ and thus the system (1) is also generalized Ulam-Hyers stable. □

To show the Ulam-Hyers-Rassias stability, we make the following hypothesis:

(H3) There exist an increasing function $\psi \in C_{1-v}[\mathcal{J}, \mathbb{R}]$ and $\lambda > 0$ such that

$$I_{0+}^\alpha\psi(s) \leq \lambda\psi(s) \quad \text{for each } s \in \mathcal{J}$$

Lemma 4.8. *Let $\xi \in \mathcal{Y}_\nu^\nu$ be a solution of the inequality (18). Then ξ satisfies the following integral inequality*

$$\left\|\xi(s) - U_{\alpha,\beta}(s)\xi_0 - \int_0^s R_\beta(s-\tau)K_\xi(\tau)d\tau\right\| \leq M\lambda\epsilon\psi(s).$$

Proof. The proof of this lemma is directly derived from Lemma 4.6 and Remark 4.5. □

Theorem 4.9. *Let us suppose that (H2), (H3) and (12) hold. Then, the system (1) is Ulam-Hyers-Rassias stable.*

Proof. Take $\varepsilon > 0$. We suppose that the function $\xi \in \mathcal{Y}_{\nu}^{\nu}$ satisfies the following inequality

$$\|D_{0+}^{\alpha,\beta}\xi(s) - A\xi(s) - \zeta(s,\xi(s),D_{0+}^{\alpha,\beta}\xi(s))\| \leq \varepsilon, \quad s \in \mathcal{J}, \tag{21}$$

and $I_{0+}^{1-\nu}\xi(0) = \xi_0$. Since all the conditions of Theorem 3.2 are satisfied, the system (1) has a unique mild solution. If we let $\varkappa \in C_{1-v}^{v}[\mathcal{J}, X]$ be the unique solution of implicit system (1), then we have

$$\varkappa(s) = U_{\alpha,\beta}(s)\varkappa_0 + \int_0^s R_\beta(s-\tau)\zeta(\tau,\varkappa(\tau),D_{0+}^{\alpha,\beta}\varkappa(\tau))d\tau.$$

We obtain from Remark 4.5 and Lemma 4.8 that

$$\left\|\xi(s) - U_{\alpha,\beta}(s)\xi_0 - \int_0^s R_\beta(s-\tau)K_\varkappa(\tau)d\tau\right\| \leq \lambda\psi(s)\varepsilon.$$

Clearly, if $\xi_0 = \varkappa_0$, then $I_{0+}^{1-v}\xi(0) = I_{0+}^{1-v}\varkappa(0)$ and

$$U_{\alpha,\beta}(s)\xi_0 = U_{\alpha,\beta}(s)\varkappa_0.$$

Using (H3) and Lemma 2.6, we obtain

$$\begin{aligned}
\|\xi(s)-\varkappa(s)\| &= \left\|\xi(s) - U_{\alpha,\beta}(s)\varkappa_0 - \int_0^s R_\beta(s-\tau)K_\varkappa(\tau)d\tau\right\| \\
&\leq \left\|\xi(s) - U_{\alpha,\beta}(s)\xi_0 - \int_0^s R_\beta(s-\tau)K_\xi(\tau)d\tau\right\| \\
&+ \left\|\int_0^s R_\beta(s-\tau)\left[K_\xi(\tau)d\tau - K_\varkappa(\tau)\right]d\tau\right\| \\
&\leq M\lambda\psi(s)\varepsilon + \frac{M}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}\left[\frac{L_1+\|A\|L_2}{1-L_2}\|\xi(\tau)-\varkappa(\tau)\|\right]d\tau \\
&\leq M\lambda\psi(s)\varepsilon + \frac{Mh_1(L_1+\|A\|L_2)}{(1-L_2)\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}M\lambda\psi(\tau)\varepsilon\Big]d\tau \\
&\leq M\lambda\left[1+\frac{Mh_1(L_1+\|A\|L_2)\lambda}{(1-L_2)\Gamma(\beta)}\right]\varepsilon\psi(s) \\
&= G_{f,\psi}\varepsilon\psi(s)
\end{aligned}$$

where $h_1 = h_1(\beta)$ is some constant. Thus, the system (1) is Ulam-Hyers-Rassias stable. □

5. Examples

In this section, we validate our results with the help of examples.

Example 5.1. *Consider Fractional differential equation*

$$\begin{cases} D_{2+}^{\frac{1}{2},\frac{1}{3}}\varkappa(s) &= -\varkappa(s) + \frac{6+|\varkappa(s)|+|D_{2+}^{\frac{1}{2},\frac{1}{3}}\varkappa(s)|}{10e^{t+3}(1+|\varkappa(s)|+|D_{2+}^{\frac{1}{2},\frac{1}{3}}\varkappa(s)|)}, \quad s \in \mathcal{J} = (2,3] \\ I_{2+}^{\frac{1}{3}}\varkappa(2) &= \varkappa_0. \end{cases} \tag{22}$$

Here $\varkappa$ is a real valued function, and $\alpha = \frac{1}{2}$, $\beta = \frac{1}{3}$ and $\nu = \frac{2}{3}$. Take

$$\zeta(s,x,y) = \frac{6+|x|+|y|}{10e^{t+3}(1+|x|+|y|)}, \quad s \in \mathcal{J} \quad \text{and } x,y \in [0,\infty).$$

Clearly ζ is continuous and

$$|\zeta(s,x,y)-\zeta(s,\bar{x},\bar{y})| \leq \frac{1}{2e^5}(|x-\bar{x}|+|y-\bar{y}|) \quad \text{for any } x,y,\bar{x},\bar{y}\in[0,\infty) \quad \text{and } s\in\mathcal{J}.$$

Thus the hypothesis (H2) is satisfied with $L_1 = L_2 = \frac{1}{2e^5}$. Since $A = -1$, we have $\|Q(s)\| \leq 1$, i.e. $M = 1$. So we have

$$\frac{MT^{\beta}(L_1+\|A\|L_2)}{(1-L_2)\Gamma(\beta)}\mathcal{B}(\beta,\nu) = \frac{2\cdot 3^{\frac{1}{3}}}{2e^5-1}\Gamma(\frac{2}{3}) < 1.$$

Therefore we conclude from Theorem 3.2 that the system (22) has the unique solution on $\mathcal{J}$. Take $\psi(s) = s$ for $s \in \mathcal{J}$. Then we have

$$\begin{aligned} I_{2+}^{\frac{1}{2}}\psi(s) =& \frac{1}{\Gamma(\frac{1}{2})}\int_2^s (s-\tau)^{-\frac{1}{2}}\tau\, d\tau \\ \leq& \frac{s}{\Gamma(\frac{1}{2})}\int_2^s (s-\tau)^{-\frac{1}{2}}\, d\tau \\ \leq& \frac{2\psi(s)}{\sqrt{\pi}}, \end{aligned}$$

Thus the hypothesis (H3) is satisfied with $\lambda = \frac{2}{\sqrt{\pi}}$. Finally, in view of Theorem 4.9, we say that the problem (22) is Ulam-Hyers-Rassias stable.

Example 5.2. *Consider the fractional differential equation of the form*

$$\begin{cases} D_{2+}^{\frac{1}{5},\frac{1}{4}}\varkappa(s) &= A\varkappa(s) + \frac{1}{e^{4s}}\left\{\begin{bmatrix}\sin(\varkappa_1(s))\\ \sin(\varkappa_2(s))\end{bmatrix} + \begin{bmatrix}\cos(D_{2+}^{\frac{1}{5},\frac{1}{4}}x_1(s))\\ \cos(D_{2+}^{\frac{1}{5},\frac{1}{4}}\varkappa_2(s))\end{bmatrix}\right\} \\ I_{2+}^{\frac{2}{5}}\varkappa(2) &= \varkappa_0. \end{cases} \tag{23}$$

where $\varkappa(s) = (\varkappa_1(s), \varkappa_2(s)) \in \mathbb{R}^2$, $\alpha = \frac{1}{5}$, $\beta = \frac{1}{4}$ and $\nu = \frac{2}{5}$,

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}. \tag{24}$$

Take

$$\zeta(s,\varkappa,y) = \frac{1}{e^{4s}}\left\{\begin{bmatrix}\sin(\varkappa_1(s))\\ \sin(\varkappa_2(s))\end{bmatrix} + \begin{bmatrix}\cos(y_1(s))\\ \cos(y_2(s))\end{bmatrix}\right\}$$

for any $\varkappa = (\varkappa_1, \varkappa_2), y = (y_1, y_2) \in \mathbb{R}^2$. Let $\mathbb{R}^2$ be a Banach space with norm $\|(\varkappa_1,\varkappa_2)\| = |\varkappa_1| + |\varkappa_2|$, $(\varkappa_1,\varkappa_2) \in \mathbb{R}^2$. For any $\varkappa, y, \bar{\varkappa}, \bar{y} \in \mathbb{R}^2$ and $s \in \mathcal{J}$, we get

$$\|\zeta(s,\varkappa,y) - \zeta(s,\bar{\varkappa},\bar{y})\| \leq \frac{1}{e^8}(\|\varkappa-\bar{\varkappa}\| + \|y-\bar{y}\|).$$

Thus the hypothesis (H2) is satisfied with $L_1 = L_2 = \frac{1}{e^8}$. It is easy to verify that $\|A\| = 2$ and $\|Q(s)\| = \|\exp(sA)\| \leq 1$, i.e. $M = 1$. So we have

$$\frac{MT^{\beta}(L_1+\|A\|L_2)}{(1-L_2)\Gamma(\beta)}\mathcal{B}(\beta,\nu) = \frac{3\cdot 3^{\frac{1}{4}}\Gamma(\frac{2}{5})}{(e^8-1)\Gamma(\frac{13}{20})} < 1.$$

Hence it follows from Theorem (3.2) that the (23) has unique solution. In addition, we conclude from Theorem (4.7) that the problem (23) is Ulam-Hyers stable.

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