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The existence of positive solutions for a Caputo-Hadamard boundary value problem with an integral boundary condition

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Abstract

In this research, by entering an $\mathfrak{a}-\theta$-Geraghty contraction, the existence of a positive solution for a boundary value problem with Caputo-Hadamard derivative including an integral boundary condition is studied. These new results improve and generalize the results stated in the literature. Some examples support and clarify our findings.

Keywords: Boundary value problem, Caputo-Hadamard derivative, integral boundary condition, $\mathfrak{a}-\theta$-Geraghty contraction.
2010 MSC: 74H10, 26A33, 34A08.

1. Introduction

Regarding the extensive applications of fractional differential equations in the modeling of many physical, chemical, medical, and engineering phenomena, this type of differential equation has attracted the attention of most scientists and mathematicians in the last few decades [3–5, 16, 21, 28–30]. Boundary and initial value problems, including fractional differential equations, have also attracted the attention of mathematicians as a result of these modelings and considerable progress has been made in studieng the existence of solutions for these types of problems [12, 14, 15, 17, 18, 32–35]. Along with examining the applications of fractional differential equations, some extensions like Hdamard derivative for fractional operators were also presented

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[24–26, 36, 37]. However, few studies have been conducted on investigating positive solutions to boundary value problems including Hadamard-type fractional differential equations.

In [23] some authors by applying some fixed point theorems studied the positive solutions of the following boundary value problem

$$\begin{aligned}
&D_1^{\beta}(D_1^{\alpha}y(x)-h(x,y_x))=g(x,y_x), \quad x\in[1,a],\ a>1,\\
&y(x)=\psi(x), \quad x\in[1-\rho,1],\\
&D_1^{\varrho}y(1)=\mu\in\mathbb{R},
\end{aligned}$$

where D_1^{β} and D_1^{α} are the Caputo-Hadamard fractional derivatives and $0<\alpha,\beta<1$.
In [13] Ardjouni and coauthors by using an upper and lower solution method and applying the Schauder and Banach fixed point theorems investigated the existence and uniqueness of positive solutions for the fractional integral boundary value problem

$$\begin{aligned}
&D_1^{\delta}u(\tau)=h(\tau,u(\tau)), \quad 1<\tau<e,\\
&u(1)=\mu\int_1^e u(\sigma)d\sigma+c,
\end{aligned}$$

where D_1^{δ} is the Caputo-Hadamard fractional derivative of order $0<\delta\leq 1,\ \mu>0, c>0, h\in C([1,e]\times[0,\infty),[0,\infty))$.
Lachouri and et al. in [20] investigated the mild solutions for the boundary value problem

$$\begin{aligned}
&{}^{CH}D_1^{\mu}y(t)-g(t,y(t)), \quad t\in[1,\tau],\\
&y(1)+y(\tau)=\lambda\int_1^{\tau}y(x)\frac{dx}{x},
\end{aligned}$$

where D_1^{μ} is the Caputo-Hadamard fractional derivative of order $0<\mu<1, \lambda\in\mathbb{R}$ with the specific properties. They applied the Schaefer's and Banach fixed point theorems to reach their purpose.
Motivated by the above works in this research the fractional integral boundary value problem

$$\begin{array}{l}
{}^{CH}D_1^{\mu}v(\tau)+h(\tau,v(\tau))=0, \quad \tau\in[1,e],\ 2<\mu\leq 3,\\
v'(1)=v''(1)=0, \quad v(e)=\eta\int_1^e v(\tau)\frac{d\tau}{\tau},
\end{array} \tag{1}$$

where $0\leq\eta<1$, ${}^{CH}D^{\mu}$ is the Caputo-hadamard fractional derivative of order μ and $h:[1,e]\times[0,\infty)\to[0,\infty)$ is a function.

2. Priliminarires

Here we recall some definitions, lemmas and theorems that will be used in this paper. One can find more details in [19, 29].

Definition 2.1. *([19]). Let $f:[1,+\infty)\to\mathbb{R}$ be a continuous function and $\varrho>0$, the Hadamard fractional integral of order ϱ is defined as*

$$^{H}\mathfrak{I}_1^{\varrho}f(t)=\frac{1}{\Gamma(\varrho)}\int_1^t\left(\log\frac{t}{s}\right)^{\varrho-1}f(s)\frac{ds}{s}.$$

Definition 2.2. *([19]). Let $f:[1,+\infty)\to\mathbb{R}$ be a continuous function and $\varrho>0$, the Caputo-Hadamard fractional derivative of order ϱ is defined as*

$$^{C\mathcal{H}}\mathfrak{D}_1^{\varrho}f(t)=\frac{1}{\Gamma(n-\varrho)}\int_1^t\left(\log\frac{t}{s}\right)^{n-\varrho-1}\left(s\frac{d}{ds}\right)^n f(s)\frac{ds}{s}.$$

Lemma 2.3. *([19]) Let $n-1<\varrho\leq n, n\in\mathbb{N}$ and $f\in C^n([1,T])$. Then*

$$({}^{H}\mathfrak{J}_1^{\varrho\mathcal{CH}}\mathfrak{D}_1^{\varrho})(f(t)) = f(t) - \sum_{j=0}^{n-1}\frac{f^j(1)}{\Gamma(j+1)}\log(t)^j.$$

Consider $d: X\times X\to\mathbb{R}^+$ given by

$$d(\nu,\omega) = \parallel \nu-\omega \parallel_\infty = \sup_{s\in\mathcal{I}}|\nu(s)-\omega(s)|, \tag{2}$$

where $X=C(\mathcal{I},\mathbb{R})$, and (X,d) is complete metric space.

3. Green Function and Bounds

Lemma 3.1. *The unique solutions of fractional boundary value problem*

$$\begin{array}{l} {}^{CH}D_1^{\mu}v(\tau)+\rho(\tau)=0, \quad \tau\in[1,e], \\ v'(1)=v''(1)=0, \quad v(e)=\eta\int_1^e v(\tau)\frac{d\tau}{\tau}, \end{array} \tag{3}$$

is given by

$$u(t) = \int_1^e G(\tau,\varsigma)\rho(\varsigma)\frac{d\varsigma}{\varsigma},$$

where

$$\mathcal{G}(\tau,\varsigma) = \begin{cases} \frac{(1-\log\varsigma)^{\mu-1}(\mu-\eta+\eta\log\varsigma)-(1-\eta)\mu(\log\tau-\log\varsigma)^{\mu-1}}{(1-\eta)\Gamma(\mu+1)} & 1\leq\varsigma\leq\tau\leq e, \\ \frac{(1-\log\varsigma)^{\mu-1}(\mu-\eta+\eta\log\varsigma)}{(1-\eta)\Gamma(\mu+1)} & 1\leq\tau\leq\varsigma\leq e. \end{cases} \tag{4}$$

Proof. By integrating of order μ from the equation of relation (3) we get

$$v(\tau) = -\frac{1}{\Gamma(\mu)}\int_1^e(\log\tau-\log\varsigma)^{\mu-1}\rho(\varsigma)\frac{d\varsigma}{\varsigma}+c_1+c_2\log\tau+c_3(\log\tau)^2, \tag{5}$$

and by differentiating above relation we have

$$v'(t) = -\frac{1}{\Gamma(\mu)}\int_1^e\frac{(\mu-1)}{\tau}(\log\tau-\log\varsigma)^{\mu-2}\rho(\varsigma)+\frac{c_2}{\tau}+\frac{2c_3}{\tau}\log\tau. \tag{6}$$

From the first boundary condition we have $v'(0)=c_2=0$, and differentiating again implies

$$\begin{aligned} v''(\tau) &= -\frac{1}{\Gamma(\mu)}\int_1^e\Big[-\frac{(\mu-1)}{\tau^2}(\log\tau-\log\varsigma)^{\mu-2} \\ &\quad +\frac{(\mu-1)(\mu-2)}{\tau^2}(\log\tau-\log\varsigma)^{\mu-3}\Big]\rho(\varsigma)\frac{d\varsigma}{\varsigma}+\frac{2c_3}{\tau^2}(1-\log\tau). \end{aligned}$$

So by the second boundary condition we get $v'(1)=2c_3=0$ or $c_3=0$. On the other hand from the third boundary condition we get

$$v(e) = -\frac{1}{\Gamma(\mu)}\int_1^e(\log e-\log\varsigma)^{\mu-1}\rho(\varsigma)\frac{d\varsigma}{\varsigma}+c_1 = \eta\int_1^e v(\varsigma)\frac{d\varsigma}{\varsigma}. \tag{7}$$

So

$$c_1 = \frac{1}{\Gamma(\mu)}\int_1^e(\log e-\log\varsigma)^{\mu-1}\rho(\varsigma)\frac{d\varsigma}{\varsigma}+\eta\int_1^e v(\varsigma)\frac{d\varsigma}{\varsigma}, \tag{8}$$

by replacing relation (8) in the relation (5) we have

$$v(t) = -\frac{1}{\Gamma(\mu)}\int_1^e (\log\tau - \log\varsigma)^{\mu-1}\rho(\varsigma)\frac{d\varsigma}{\varsigma} \tag{9}$$

$$\frac{1}{\Gamma(\mu)}\int_1^e (\log e - \log\varsigma)^{\mu-1}\rho(\varsigma)\frac{d\varsigma}{\varsigma} + \eta\int_1^e v(\varsigma)\frac{d\varsigma}{\varsigma}. \tag{10}$$

Now by integrating of the relation (10) we have

$$\begin{aligned}\int_1^e v(\tau)\frac{d\tau}{tau} &= -\frac{1}{\Gamma(\mu)}\int_1^e\int_1^\tau (\log\tau - \log\varsigma)^{\mu-1}\rho(\varsigma)\frac{d\varsigma}{\varsigma}\frac{d\tau}{\tau} \\ &\quad +\frac{1}{\Gamma(\mu)}\int_1^e\int_1^e (\log\tau - \log\varsigma)^{\alpha-1}\rho(\varsigma)\frac{d\varsigma}{\varsigma}\frac{d\tau}{\tau} + \eta\log e\int_1^e v(\varsigma)\frac{d\varsigma}{\varsigma}. \end{aligned}\tag{11}$$

So

$$\begin{aligned}(1-\eta)\int_1^e v(\varsigma)\frac{d\varsigma}{\varsigma} &= -\frac{1}{\mu(\Gamma(\mu)}\int_1^e (\log\tau - \log\varsigma)^{\mu}\rho(\varsigma)\frac{d\varsigma}{\varsigma} \\ &\quad +\frac{1}{\Gamma(\mu)}\int_1^e (\log e - \log\varsigma)^{\mu-1}\rho(\varsigma)\frac{d\varsigma}{\varsigma},\end{aligned}$$

or

$$\begin{aligned}\int_1^e v(\varsigma)\frac{d\varsigma}{\varsigma} &= -\frac{1}{\mu(\Gamma(\mu)(1-\eta)}\int_1^e (\log\tau - \log\varsigma)^{\mu}\rho(\varsigma)\frac{d\varsigma}{\varsigma} \\ &= +\frac{1}{\Gamma(\mu)(1-\eta)}\int_1^e (\log e - \log\varsigma)^{\mu-1}\rho(\varsigma)\frac{d\varsigma}{\varsigma},\end{aligned}$$

Consequently

$$\begin{aligned}v(t) &= -\frac{1}{\Gamma(\mu)}\int_1^e (\log\tau - \log\varsigma)^{\mu-1}\rho(\varsigma)\frac{d\varsigma}{\varsigma} \\ &\quad \frac{1}{\Gamma(\mu)}\int_1^e (\log e - \log\varsigma)^{\mu-1}\rho(\varsigma)\frac{d\varsigma}{\varsigma} \\ &\quad -\frac{\eta}{\mu(\Gamma(\mu)(1-\eta)}\int_1^e (\log e - \log\varsigma)^{\mu}\rho(\varsigma)\frac{d\varsigma}{\varsigma} \\ &= +\frac{\eta}{\Gamma(\mu)(1-\eta)}\int_1^e (\log e - \log\varsigma)^{\mu-1}\rho(\varsigma)\frac{d\varsigma}{\varsigma} \\ &= -\frac{\mu(1-\eta)}{(1-\eta)\Gamma(\mu+1)}\int_1^e (\log\tau - \log\varsigma)^{\mu-1}\rho(\varsigma)\frac{d\varsigma}{\varsigma} \\ &\quad +\frac{1}{(1-\eta)\Gamma(\mu+1)}\int_1^e (\mu - \eta\log e + \log\varsigma)(\log\tau - \log\varsigma)^{\mu-1}d\rho(\varsigma)\frac{d\varsigma}{\varsigma} \\ &= \int_1^e \mathcal{G}(\tau,\varsigma)\rho(\varsigma)\frac{d\varsigma}{\varsigma}\end{aligned}$$

□

Lemma 3.2. *Let $\mathcal{G}(\tau,\varsigma)$ is defined by the relation (4) and $0 \le \eta < 1$. Then the following conditions are hold.*

1. $\mathcal{G}(\tau,\varsigma) \ge 0$ *for all* $1 \le, \varsigma, \tau \le e$;
2. $\mathcal{G}(\tau,\varsigma)$ *is a continuous in* $[1,e]\times[1,e]$;
3. $\mathcal{G}(e,\varsigma) \le \mathcal{G}(\tau,\varsigma) \le \frac{\mu}{\eta(\mu-1)}\mathcal{G}(e,\varsigma)$ *for all* $1 \le, \varsigma, \tau \le e$;
4. $\max_{\tau,\varsigma\in[1,e]}\{\mathcal{G}(\tau,\varsigma)\} \le \frac{1}{(1-\eta)\Gamma(\mu)}$.

Proof. 1. For $\varsigma \leq \tau$, we have

$$\begin{aligned} G(\tau,\varsigma) =& \frac{(1-\log\varsigma)^{\mu-1}(\mu-\eta+\eta\log\varsigma)-(1-\eta)\mu(\log\tau-\log\varsigma)^{\mu-1}}{(1-\eta)\Gamma(\mu+1)} \\ \geq& \frac{(1-\log\varsigma)^{\mu-1}(\mu-\eta+\eta\log\varsigma)-(1-\eta)\mu(1-\log\varsigma)^{\mu-1}}{(1-\eta)\Gamma(\mu+1)} \\ =& \frac{(1-\log\varsigma)^{\mu-1}\eta(\mu+\log\varsigma-1)}{(1-\eta)\Gamma(\mu+1)} \geq 0. \end{aligned}$$

For $\tau \leq \varsigma$, it is clear that

$$G(\tau,\varsigma) = \frac{(1-\log\varsigma)^{\mu-1}(\mu-\eta+\eta\log\varsigma)}{(1-\eta)\Gamma(\mu+1)} \geq 0.$$

2. It is concluded directly from the definition of the function G.
3. For $s \leq \tau$, we have

$$\frac{G(\tau,\varsigma)}{G(e,\varsigma)} = \frac{(1-\log\tau)^{\mu-1}(\mu-\eta+\eta\log\varsigma)-(1-\eta)\mu(1-\log\varsigma)^{\mu-1}}{\eta(\log\varsigma+\mu-1)(1-\log\varsigma)^{\mu-1}} = 1,$$

and for $0 \leq \tau \leq \varsigma \leq e$, we get

$$1 \leq \frac{\log\varsigma - 1 + \frac{\mu}{\eta}}{\log\varsigma - 1 + \mu} \leq \frac{G(\tau,\varsigma)}{G(e,\varsigma)} \leq \frac{\mu}{\eta(\mu-1)}.$$

4. For $\tau, \varsigma \in [1,e]$, we conclude that

$$G(\tau,\varsigma) \leq \frac{(1-\log\varsigma)^{\mu-1}(\mu-\eta+\eta\log\varsigma)}{(1-\mu)\Gamma(\mu+1)} \leq \frac{1}{(1-\mu)\Gamma(\mu)}.$$

□

Let Θ contains all $\theta : \mathbb{R}^+ \to \mathbb{R}^+$ (θ is increasing) with $\sum_{n=1}^{+\infty} \theta^n(\omega) < \infty$, $\omega > 0$.

Definition 3.3. *[1, 2] A function $g : X \to X$ ((X,d) is complete) is said to be an $\mathfrak{a}$-θ contraction if there exists $\mathfrak{a} : X \times X \to \mathbb{R}^+$ with*

$$\mathfrak{a}(\nu,\omega)\theta\big(d(g\nu,g\omega)\big) \leq \theta\big(d(\nu,\omega)\big),\ \nu,\omega \in X,\ \theta \in \Theta.$$

Definition 3.4. *[10] Let $g : X \to X$ and $\mathfrak{a} : X \times X \to \mathbb{R}^+$ be given. Then g is called $\mathfrak{a}$-admissible if for $\nu,\omega \in X$,*

$$\mathfrak{a}(\nu,\omega) \geq 1 \Rightarrow \mathfrak{a}(g\nu,g\omega) \geq 1.$$

Theorem 3.5. *[10] Let (X,d) be a complete metric space and $\varphi : X \to X$ be a $\mathfrak{a}-\theta$ contraction such that*
(i) φ is $\mathfrak{a}$-admissible;
(ii) $\exists\ w_0 \in X$ with $\mathfrak{a}(w_0,\varphi w_0) \geq 1$;
(iii) $\{w_n\} \subseteq X$, $w_n \to u$ in X and $\mathfrak{a}(w_n,w_{n+1}) \geq 1$ then $\mathfrak{a}(w_n,w) \geq 1$.
Then φ has a fixed point.

4. Existence Results

Theorem 4.1. *Suppose*
There exist $\rho : \mathbb{R}^2 \to \mathbb{R}$ *and* $\theta \in \Theta$ *with the following property:*

(i) $\big|h(\tau, ß(\omega)) - h(\tau, \pi(\omega))\big| \leq \frac{(1-\eta)\Gamma(\mu)}{e-1}\theta(ß(\omega) - \pi(\omega))$. *for* $\omega \in \mathcal{I}$ *and* $ß, \pi \in X$*;*

(ii) there exists $ß_0 \in C(\mathcal{I})$ *such that*

$$\rho\big(ß_0(\rho)\big), \int_1^e G(\tau,\eta)h\big(\eta, ß_0(\tau)\big)d\eta) \geq 0 \qquad \tau \in \mathcal{I};$$

(iii) If $\rho(ß(\tau), \pi(\tau)) \geq 0$*, then*

$$\rho\Big(\int_1^e G(\tau,\eta)h\big(\eta, ß(\eta)\big)d_\eta, \int_1^e G(\tau,\eta)h\big(\eta, \pi(\tau)\big)d\eta\Big) \geq 0;$$

(iv) if $\{ß_n\} \subseteq C(\mathcal{I})$*,* $ß_n \to ß$ *in* $C(\mathcal{I})$*, and* $\rho(ß_n, ß_{n+1}) \geq 0$*, then* $\rho(ß_n, ß) \geq 0$*.*

Then Problem (1) *has at least one solution.*

Proof. From Lemma 3.1, $ß \in C(\mathcal{I})$ is a solution of (1) if and only if is a solution of

$$ß(\tau) = \int_1^e G(\tau,\omega)h\big(\omega, ß(\omega)\big)ds.$$

Thus we find the fixed point of the operator $\digamma : C(\mathcal{I}) \to C(\mathcal{I})$ defined by

$$\digamma ß(\tau) = \int_1^e G(\tau,\omega)h\big(\omega, ß(\omega)\big)d\omega.$$

Let $ß, \pi \in C(\mathcal{I})$ with $\rho(ß(\tau), \pi(\tau)) \geq 0$. By (i), we obtain

$$\begin{aligned}
\big|\digamma ß(\tau) - \digamma \pi(\tau)\big| &= \Big|\int_1^e G(\tau,\omega)h\big(\omega, ß(\omega)\big)d\omega - \int_1^e G(\tau,\omega)h\big(\omega, \pi(\omega)\big)d\omega\Big| \\
&= \Big|\int_1^e G(\tau,\omega)\Big(h\big(\omega, ß(\omega)\big) - h\big(\omega, \pi(\omega)\big)\Big)d\omega\Big| \\
&\leq \Big[\int_1^e G(\tau,\omega)\Big|h\big(\omega, ß(\omega)\big) - h\big(\omega, \pi(\omega)\big)\Big|d\omega\Big] \\
&\leq \Big[\int_1^e \frac{1}{(1-\eta)\Gamma(\mu)} \\
&\times \frac{(1-\eta)\Gamma(\mu)}{e-1}\int_1^e \theta(ß(\omega) - \pi(\omega))d\omega \\
&\leq \theta(\|ß(\omega) - \pi(\omega)\|_\infty).
\end{aligned}$$

So, we have $\big\|(\digamma ß - \digamma \pi)\big\| \leq \theta(\|ß(\omega) - \pi(\omega)\|_\infty)$. Let $\mathfrak{a} : C(\mathcal{I}) \times C(\mathcal{I}) \to \mathbb{R}^+$ be defined by

$$\mathfrak{a}(ß, \pi) = \begin{cases} 1 & \rho\big(ß(\tau), \pi(\tau)\big) \geq 0, \quad \tau \in \mathcal{I}, \\ 0 & \text{otherwise.} \end{cases}$$

We have;

$$\mathfrak{a}(ß, \pi)d(\digamma ß, \digamma \pi) \leq \mathfrak{a}(ß, \pi)\theta(d(ß, \pi)).$$

Then $\digamma$ is an $\mathfrak{a}$-θ-contractive. From (iii) and the definition of $\mathfrak{a}$ we get

$$\begin{aligned}\mathfrak{a}(\text{ß},\pi)\geq 1 &\Rightarrow \rho\big(\text{ß}(\tau),\pi(\tau)\big)\geq 0\\ &\Rightarrow \rho\big(\digamma(\text{ß}),\digamma(\pi)\big)\geq 0\\ &\Rightarrow \mathfrak{a}\big(\digamma(\text{ß}),\digamma(\pi)\big)\geq 1,\end{aligned}$$

for $\text{ß},\pi\in C(\mathcal{I})$. Thus, $\digamma$ is $\mathfrak{a}$-admissible. By (ii) $\exists\ \text{ß}_0\in C(\mathcal{I})$ with $\mathfrak{a}(\text{ß}_0,\digamma\text{ß}_0)\geq 1$. From (iv) and Theorem 4.1, there is $\text{ß}^*\in C(\mathcal{I})$ with $\text{ß}^*=\digamma\text{ß}^*$. Hence ß^* is a solution of the problem. □

Example 4.2. *Let* $\theta(r)=\frac{r}{2}$, $\rho(y,z)=yz$, $\eta_n(\omega)=\dfrac{\omega}{n^2+1}$.
Consider $h:\mathcal{I}\times C(\mathcal{I})\to[0,\infty]$ *and the boundary value problem*

$$^{CH}D^{\frac{5}{2}}v(\omega)+\frac{3\sqrt{\pi}}{32}\cos\omega^2 v(\omega)=0, \tag{12}$$

with $v'(0)=v''(0)=0,\ v(1)=\frac{1}{2}\int_0^1 v(s)ds$,
also, $h(\omega,v(\omega))=\frac{3\sqrt{\pi}}{32}\cos\omega^2 v(\omega)$, $(\omega,v(\omega))\in\mathcal{I}\times\mathbb{R}^+$.
Since $\theta^n(t)\to 0$ *when* $n\to 0$, *hence* $\theta\in\Theta$.

Also,

$$\begin{aligned}&|h(\omega,y(\omega))-h(\omega,z(\omega))|\leq\frac{3\sqrt{\pi}}{32}\big|\cos\omega^2 y(\omega)-\cos\omega^2 z(\omega)\big|\\ &=\frac{3\sqrt{\pi}}{32}\Big|-2\sin\frac{\omega^2 y(\omega)-\omega^2 z(\omega)}{2}\sin\frac{\omega^2 y(\omega)+\omega^2 z(\omega)}{2}\Big|\\ &\leq\frac{3\sqrt{\pi}}{16(e-1)}\frac{\omega^2|y(\omega)-z(\omega)|}{2}\\ &\leq\frac{(1-\eta)\Gamma(\mu)}{e-1}\theta(|y(\omega)-z(\omega)|),\end{aligned}$$

when $\omega\in\mathcal{I}$ *and* $y(\omega),z(\omega)\in\mathbb{R}^+$ *with* $\rho(y(\omega),z(\omega))\geq 0$. *Hence,*

$$|h(\omega,y(\omega))-h(\omega,z(\omega))|\leq\frac{(1-\eta)\Gamma(\mu)}{e-1}\theta(|y(\omega)-z(\omega)|).$$

So the condition (i) from Theorem (4.1) hold.

If $y_0(\omega)=\omega$, *then*

$$\rho\Big(y_0(\omega),\int_0^1 G(\omega,\omega)\omega f(\omega,y_0(\omega))d\omega\Big)\geq 0.$$

for $\omega\in\mathcal{I}$. *Also,*
$\rho(y(\omega),z(\omega))=y(\omega)z(\omega)\geq 0$ *implies that*

$$\rho\Big(\int_0^1 G(\omega,\omega)\omega f(\omega,y_(\omega))d\omega,\int_0^1 G(\omega,\omega)\omega f(\omega,z(\omega))d\omega\Big)\geq 0.$$

It is obviously that condition (iv) in Theorem (4.1) hold. Hence, the all conditions (4.1) satisfied. So the equation (1) *has at least one solution.*

Suppose $\mathbf{F}$ consists of all functions $\wp:(0,\infty)\to R$ such that
$(\mathbf{F}_1)$ $0<\omega<t\ \Rightarrow\ \wp(\omega)\leq\wp(t)$;
$(\mathbf{F}_2)$ $s_n\to 0\Leftrightarrow\wp(s_n)\to-\infty$,

In this case, we can state some of our results as follows:

Definition 4.3. *[27] Let $\wp \in \mathbf{F}$, $a \in \mathbb{R}^+$ and $d : \mathfrak{M} \times \mathfrak{M} \to [0, +\infty)$ such that*
(e_1) $(\phi, \sigma) \in \mathfrak{M} \times \mathfrak{M}, d(\phi, \sigma) = 0 \Leftrightarrow \phi = \sigma$;
(e_2) $d(\phi, \sigma) = d(\sigma, \phi)$, for $(\phi, \sigma) \in \mathfrak{M} \times \mathfrak{M}$;
(e_3) If $(\phi, \sigma) \in \mathfrak{M} \times \mathfrak{M}$, $(u_i)_{i=1}^N \subset \mathfrak{M}$ with $(u_1, u_N) = (\phi, \sigma)$, $N \in \mathrm{N}, N \geq 2$ we obtain

$$d(\phi, \sigma) > 0 \Rightarrow \wp(d(\phi, \sigma)) \leq \wp\left(\sum_{i=1}^{N-1} d(u_i, u_{i+1})\right) + a.$$

Then d is an $\mathbf{F}$-metric on $\mathfrak{M}$, and $(\mathfrak{M}, d)$ is an $\mathbf{F}$-metric space.

The convergence, condition of Cauchy sequences, and completeness are those defined in the normal metric space.

Definition 4.4. *[9] If $\wp : \mathfrak{M} \to \mathfrak{M}$, $\mathfrak{a} : \mathfrak{M} \times \mathfrak{M} \to \mathbb{R}^+$, and*

$$\mathfrak{a}(\omega, \wp\omega) \geq 1 \Rightarrow \mathfrak{a}(\wp\omega, \wp^2\omega) \geq 1. \tag{13}$$

Then $\wp$ is $\mathfrak{a}-$orbital admissible.

Theorem 4.5. *[22] Let $\wp : \mathfrak{M} \to \mathfrak{M}$ ($(\mathfrak{M}, d)$ is a $\mathbf{F}$-complete) such that*

$$\mathfrak{a}(\phi, \sigma) d(\wp\phi, \wp\sigma) \leq \ell(d(\phi, \sigma)),$$

for $\phi, \sigma \in \mathfrak{M}$, $\ell \in \Theta$. Also suppose

(a_1) $\wp$ is $\mathfrak{a}$-orbital admissible;

(a_2) exists $\phi_0 \in \mathfrak{M}$ with $\mathfrak{a}(\phi_0, \wp\phi_0) \geq 1$;

(a_3) $\wp \in \mathbf{F}$ which also has property (e_3) is continuous and if ℓ is continuous then we get $\wp(u) > \wp(\ell(u)) + a, 0 < u < \infty$, where a is the same stated in (e_3);

Then f a fixed point.

Consider $d : \mathfrak{M} \times \mathfrak{M} \to \mathbb{R}^+$ as

$$d(\phi, \sigma) = \begin{cases} e^{\|\phi - \sigma\|_\infty} & \text{if } \phi \neq \sigma \\ 0 & \text{if } \phi = \sigma, \end{cases}$$

with $\mathfrak{M} = \mathcal{C}(\mathcal{I}, \mathrm{N})$. Define $\wp$ on $(0, \infty)$ by $\wp(\omega) = -\frac{1}{\omega}, \omega > 0$. Since $-\frac{1}{u} > \frac{1}{\ell(u)} > 1$, clearly $\wp(u) > \wp(\ell(u)) + a$, $u > 0$, indeed ℓ has the following properties

$$\ell(u) < \frac{u}{u+1},$$

$$e^{\ell(\omega)} \leq \ell(e^\omega), \ \omega \in \{0, 1, 2, 3, \ldots\}.$$

Theorem 4.6. *Suppose there exists $\jmath : R^2 \to R$ such that*
(a_1)

$$|h(\omega, \phi(\omega)) - h(\omega, \sigma(\omega))| \leq \frac{(1-\eta)\Gamma(\mu)}{e-1} \ell\left(|\phi(\omega) - \sigma(\omega)|\right),$$

where $s \in \mathcal{I}$ and $\phi, \sigma \in R$ with $\jmath(\phi, \sigma) \geq 0$;
(a_2) there exist $\hbar_1 \in \mathcal{C}(\mathcal{I})$ with $\jmath(\phi_1(\omega), \digamma\phi_1(\omega)) \geq 0$, $\omega \in \mathcal{I}$ and $\digamma : \mathcal{C}(\mathcal{I}) \to \mathcal{C}(\mathcal{I})$ is given by

$$(\digamma\phi)(p) = \int_1^e G(\tau, \omega) h(\omega, \phi(\omega)); \tag{14}$$

(a_3) for $\omega \in \mathcal{I}$ and $\phi \in \mathcal{C}(\mathcal{I})$, $\jmath(\phi(\omega), \digamma\phi(\omega)) \geq 0$ implies $\jmath(\digamma\phi(\omega), \digamma^2\phi(\omega)) \geq 0$;
Then $\digamma$ has a fixed point and it is a solution of (1).

Proof. Using a proof similar to the proof of Theorem 4.1, we conclude that

$$|\digamma\phi(\tau)-\digamma\sigma(\tau)| \leq \ell\left(\|\phi(\omega)-\sigma(\omega)\|_{\infty}\right).$$

So for $\phi,\sigma\in\mathcal{C}(\mathcal{I})$, $\omega\in\mathcal{I}$ with $\jmath(\phi(\omega),\sigma(\omega))\geq 0$, we get

$$d(\digamma\phi,\digamma\sigma)=e^{\|\digamma\phi-\digamma\sigma\|_{\infty}}\leq e^{\ell\left(\|\phi(\omega)-\sigma(\omega)\|_{\infty}\right)}\leq \ell(e^{\|\phi(\omega)-\sigma(\omega)\|_{\infty}}=\ell(d(\phi,\sigma)).$$

Put, $\mathfrak{a}:\mathcal{C}(\mathcal{I})\times\mathcal{C}(\mathcal{I})\to\mathbb{R}^{+}$ by

$$\mathfrak{a}(\phi,\sigma)=\begin{cases}1 & \jmath(\phi(\omega),\sigma(\omega))\geq 0,\omega\in\mathcal{I},\\ 0 & else.\end{cases}$$

Therefore,
$\mathfrak{a}(\phi,\sigma)d(\digamma\phi,\digamma\sigma)\leq d(\digamma\phi,\digamma\sigma))\leq\ell(d(\phi,\sigma))$, $\phi,\sigma\in M$, with $d(\digamma\phi,\digamma\sigma)>0$.

From (a_3),

$$\mathfrak{a}(\phi,\digamma\phi)\geq 1\Rightarrow\jmath(\phi(\omega),\digamma\phi(\omega))\geq 0\Rightarrow\jmath(\digamma(\phi),\digamma^{2}(\phi))\geq 0\Rightarrow\mathfrak{a}(\digamma(\phi),\digamma^{2}(\phi))\geq 1,$$

for $\phi\in\mathcal{C}(\mathcal{I})$. Hence $\digamma$ is $\mathfrak{a}-$ orbital admissible. From (a_2), put $\phi_1\in\mathcal{C}(\mathcal{I})$ such that $\mathfrak{a}(\phi_1,\digamma\phi_1)\geq 1$. By (a_3) and Theorem 4.5, we obtain $\phi^*\in\mathcal{C}(\mathcal{I})$ with $\phi^*=\digamma\phi^*$. Hence, we obtained ϕ^* as a solution of the problem. □

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