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Solutions of Mixed Integral Equations via Hybrid Contractions

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Abstract

This paper establishes certain new fixed point results for a class of contractions known as admissible hybrid (θ-ζ)-contraction within the context of rectangular metric space. The main contribution of this work is a straightforward unification of the notions of admissible mappings, θ-contractions, and the contraction mapping principle. As a result, several corollaries are inferred from the primary findings given here, some of which comprise some previously disclosed concepts. An application to one of the obtained results is the proposal of new criteria for the existence and uniqueness of a solution to a mixed nonlinear fixed point problem, using Volterra-Fredholm integrals. Nontrivial analytical and numerical examples are given and compared with the specific articles supporting this study in order to elucidate the underlying theoretical ideas.

Keywords: fixed point, metric space θ-contraction nonlinear integral equation.
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1. Introduction

Differential equations, game theory, dynamical systems, statistical models, and real-world issues are a few examples of mathematical models. The presence of a solution for these concerns has been investigated in a number of domains of mathematics, for example, differential equations, integral equations, functional

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analysis, etc. One of these techniques for solving these issues is the fixed point (FP) approach. As a result, this approach is widely used in computer science, physics, biology, chemistry, and economics in addition to mathematics.

A widely used method for solving nonlinear analytic problems is the Banach contraction concept. This idea first surfaced in Banach's thesis [5], where it was applied to demonstrate the uniqueness and existence of integral equation solutions. It said as follows: A mapping Υ on a metric space (MS) (Υ, ϱ) is called a Banach contraction if there exists $r \in (0,1)$ such that for all $\varsigma, \varpi \in \Xi$,

$$\varrho(\Upsilon\varsigma, \Upsilon\varpi) \leq r\varrho(\varsigma, \varpi). \tag{1}$$

Notice that the contractive condition (1) is fulfilled for all $\varsigma, \varpi \in \Xi$ which forces the mapping Υ to be continuous, while it is not applicable in case of discontinuity. In view of the applicability of contraction principle, forcing the concerned mapping to be continuous remains one of its drawbacks. Many authors attempted to overcome this drawbacks, see ([32, 20]).

Branciari [7] in 2000 brought up the concept of generalized metric space (GMS), where the triangle inequality is replaced by the inequality $\varrho(\varsigma, \varpi) \leq \varrho(\varsigma, u) + \varrho(u, v) + \varrho(v, \varpi)$ for all pairwise distinct points $\varsigma, \varpi, u, v \in \Xi$. Since then, various FP results have been established on such spaces. Kirk and Shahzad [18] said a 'GMS' is a semimetric space which does not satisfy the triangle inequality, but satisfies a weaker assumption. Samet [25] also talked about exposing incorrect property of the GMS as brought in by Branciari. For other discussions, see ([3, 27]), and the references therein.

There has been an explosion in the number of articles in metric FP theory over the previous few decades. This reality compels researchers to devise a suitable method for consolidating and integrating the current findings. The concept of Kannan-type interpolative contraction, which maximizes the rate of convergence, was presented by Karapinar [15], who also brought together a number of previously published results. The concept in [15] was expanded upon by Yelsilkaya [31] to create the Hardy-Rogers contractive of the Suzuki-type mapping. Mitrovi ́c et al. [19] brought in and studied a hybrid contraction that combines a Reich-type contraction and interpolative-type contractions very recently, motivated by the result in [15]. Accordingly, Karapinar and Fulga [16] created a new hybrid contraction by fusing interpolative-type contraction with Jaggi-type contraction.

There has been little or no research on hybrid FP outcomes in relation to θ-contraction, according to the available literature search. As such, we aim to introduce admissible hybrid version of (θ-ζ)-contraction and establish various FP results for such mappings in the setting of complete GMS, inspired by the idea of Jleli and Samet [12] and the work of Karapinar and Fulga [17].

2. Preliminaries

Some basic definitions, vocabulary, and notations that will be used later are reviewed in this section. In this study, each set Ξ is regarded as non-empty; the set of all natural numbers is denoted by $\mathbb{N}$ and the set of non-negative real numbers by $\mathbb{R}_+$. A novel kind of contraction was brought in in 1969 by Kannan [13]. It is an affirmative response to the question of whether a discontinuous mapping exists in the frame of complete metric space (CMS) that satisfies specific contractive requirements and has a FP. The theorem is as follows:

Theorem 2.1. *Let (Ξ, ϱ) be a CMS and $\Upsilon : \Xi \longrightarrow \Xi$ be a mapping fulfilling*

$$\varrho(\Upsilon\varsigma, \Upsilon\varpi) \leq \lambda[\varrho(\varsigma, \Upsilon\varsigma) + \varrho(\varpi, \Upsilon\varpi)]$$

for all $\varsigma, \varpi \in \Xi$, where $\lambda \in [0, \frac{1}{2})$. Then Υ has a unique FP.

Definition 2.2. *[15] Let (Ξ, ϱ) be a MS. A self mapping Υ on Ξ is called an interpolative Kannan-type contraction if there exist a constant $\lambda \in [0,1)$ and $\alpha \in (0,1)$ such that*

$$\varrho(\Upsilon\varsigma, \Upsilon\varpi) \leq \lambda(\varrho(\varsigma, \Upsilon\varsigma))^{\alpha}.(\varrho(\varpi, \Upsilon\varpi))^{1-\alpha}$$

for all $\varsigma, \varpi \in \Xi$ with $\varsigma \neq \Upsilon\varsigma$.

In 1977, Jaggi[11] defined a new concept of a generalized Banach contraction now called Jaggi contraction, which is one of the first known rational contractive inequalities.

Definition 2.3. *[9] Let (Ξ, ϱ) be a MS. A continuous self-mapping $\Upsilon : \Xi \longrightarrow \Xi$ is called Jaggi contraction if:*

$$\varrho(\Upsilon\varsigma, \Upsilon\varpi) \leq \alpha_1 \frac{\varrho(\varsigma, \Upsilon\varsigma).\varrho(\varpi, \Upsilon\varpi)}{\varrho(\varsigma, \varpi)} + \alpha_2 \varrho(\varsigma, \varpi),$$

for all $\varsigma, \varpi \in \Xi$, $\varsigma \neq \varpi$ and for some $\alpha_1, \alpha_2 \in [0, 1)$ with $\alpha_1 + \alpha_2 < 1$.

Bianciari [7] brought in the concept of GMS where the triangular inequality is replaced with rectangular inequality and the Cauchy condition is slightly different . The definitions are as follows:

Definition 2.4. *Let Ξ be a non-empty set and $\varrho : \Xi \times \Xi \longrightarrow \mathbb{R}_+$ be a mapping such that for all $\varsigma, \varpi \in \Xi$ and for all distinct points $u, v \in \Xi$, each of them different from ς and ϖ, we have*

(i) $\varrho(\varsigma, \varpi) = 0 \Leftrightarrow \varsigma = \varpi$;

(ii) $\varrho(\varsigma, \varpi) = \varrho(\varpi, \varsigma)$;

(iii) $\varrho(\varsigma, \varpi) \leq \varrho(\varsigma, u) + \varrho(u, v) + \varrho(v, \varpi)$.

Then, (Ξ, ϱ) is called a GMS.

Recently, Jleli and Samet [12] brought in a new type of contraction called θ-contraction and established some new FP theorems for such contraction in the context of generalized metric spaces, as brought in by Bianciari [7].

Definition 2.5. *Let (Ξ, ϱ) be a GMS. A mapping $\Upsilon : \Xi \longrightarrow \Xi$ is called θ-contraction if there exists $\theta \in \Theta$ and $r \in (0, 1)$ such that for all $\varsigma, \varpi \in \Xi$,*

$$\varrho(\Upsilon\varsigma, \Upsilon\varpi) \neq 0 \Rightarrow \theta(\varrho(\Upsilon\varsigma, \Upsilon\varpi)) \leq [\theta(\varrho(\varsigma, \varpi))]^r,$$

where Θ is the set of functions $\theta : (0, \infty) \longrightarrow (1, \infty)$ fulfilling the following conditions:

(Θ_1) θ is non-decreasing;

(Θ_2) for each sequence $\{t_n\} \subset (0, \infty)$, $\lim_{n\to\infty} \theta(t_n) = 1$ if and only if $\lim_{t\to\infty} (t_n) = 0^+$;

(Θ_3) there exist $r \in (0, 1)$ and $l \in (0, \infty]$ such that $\lim_{r\to 0^+} \frac{\theta(t) - 1}{t^r} = \ell$.

They showed that the Banach contraction is a special kind of θ-contraction while there are θ-contractions which are not Banach contractions. For a recent survey of FP results of θ-contraction, we refer to [21].

Definition 2.6. *[26] Let $\Upsilon : \Xi \longrightarrow \Xi$ and $\alpha : \Xi \times \Xi \longrightarrow \mathbb{R}_+$ be mappings. Then, Υ is called α-admissible if for all $\varsigma, \varpi \in \Xi$, $\alpha(\varsigma, \varpi) \geq 1 \Rightarrow \alpha(\Upsilon\varsigma, \Upsilon\varpi) \geq 1$.*

Definition 2.7. *[24] Let $\Upsilon : \Xi \longrightarrow \Xi$ and $\alpha : \Xi \times \Xi \longrightarrow \mathbb{R}_+$ be mappings. Then, Υ is called triangular α-admissible if*

T1) Υ *is* α*-admissible,*

(T2) $\alpha(\varsigma, z) \geq 1$ *and* $\alpha(z, \varpi) \geq 1 \Rightarrow \alpha(\varsigma, \varpi) \geq 1$.

As a modification in the concept of α-admissible mappings, Popescu [23] brought in α-orbital admissible mappings as follows:

Definition 2.8. *Let $\Upsilon : \Xi \longrightarrow \Xi$ be a mapping and let $\alpha : \Xi \times \Xi \longrightarrow \mathbb{R}_+$ be a function. Υ is said to be α-orbital admissible if for all $\varsigma \in \Xi$, $\alpha(\varsigma, \Upsilon\varsigma) \geq 1$ implies $\alpha(\Upsilon\varsigma, \Upsilon^2\varsigma) \geq 1$.*

Definition 2.9. *Let $\alpha : \Xi \times \Xi \longrightarrow \mathbb{R}_+$ and $\Upsilon : \Xi \longrightarrow \Xi$ be mappings. Then, Υ is called triangular α-orbital admissible if for all $\varsigma, \varpi \in \Xi$,*

(i) Υ is α-orbital admissible;

(ii) $\alpha(\varsigma, \varpi) \geq 1$ and $\alpha(\varpi, \Upsilon\varpi) \geq 1$ implies $\alpha(\varsigma, \Upsilon\varpi) \geq 1$.

Definition 2.10. *[2] A mapping $\zeta : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is called a (c)-comparison function if it satisfies the following conditions:*

(a) ζ is nondecreasing;

(b) the series $\sum_{n=1}^{\infty} \zeta^n(z)$ is convergent for $z \geq 0$.

Lemma 2.11. *[6] Let Φ be the family of (c)-comparison functions and $\zeta \in \Phi$. Then, the following conditions hold:*

(i) $\zeta^n(z) \longrightarrow 0$ as $n \longrightarrow \infty$ for all $z \geq 0$;

(ii) $\zeta(z) < z$ for all $z > 0$;

(iii) ζ is continuous;

(iv) $\zeta(z) = 0$ if and only if $z = 0$;

(v) the series $\sum_{n=1}^{\infty} \zeta^n(z) \geq 0$.

Definition 2.12. *[17] Let (Ξ, ϱ) be a MS. A mapping $\Upsilon : \Xi \longrightarrow \Xi$ is said to be an admissible hybrid contraction if there exist $\zeta \in \Phi$, and a mapping $\alpha : \Xi \times \Xi \longrightarrow \mathbb{R}_+$ such that*

$$\alpha(\varsigma, \varpi)\varrho(\Upsilon\varsigma, \Upsilon\varpi)) \leq \zeta(M_{\lambda_i}(\varsigma, \varpi, s, \Upsilon)), \tag{2}$$

where $s \geq 0$ and $\lambda_i \geq 0$; $i = 1, 2, \ldots, 5$, such that $\displaystyle\sum_{i=1}^{5} \lambda_i = 1$ and

$$M_{\lambda_i}(\varsigma, \varpi, s, \Upsilon) = \begin{cases} \Bigg[\lambda_1(\varrho(\varsigma,\varpi))^s + \lambda_2(\varrho(\varsigma,\Upsilon\varsigma))^s + \lambda_3(\varrho(\varpi,\Upsilon\varpi))^s \\ +\lambda_4\left(\frac{\varrho(\varpi,\Upsilon\varpi)(1+\varrho(\varsigma,\Upsilon\varsigma))}{1+\varrho(\varsigma,\varpi)}\right)^s + \lambda_5\left(\frac{\varrho(\varpi,\Upsilon\varsigma)(1+\varrho(\varsigma,\Upsilon\varpi))}{1+\varrho(\varsigma,\varpi)}\right)^s\Bigg]^{\frac{1}{s}} \\ \qquad\qquad \text{for some } s > 0,\ \varsigma, \varpi \in \Xi; \\ (\varrho(\varsigma,\varpi))^{\lambda_1}.(\varrho(\varsigma,\Upsilon\varsigma))^{\lambda_2}.(\varrho(\varpi,\Upsilon\varpi))^{\lambda_3}. \\ \left(\frac{\varrho(\varpi,\Upsilon\varpi)(1+\varrho(\varsigma,\Upsilon\varsigma))}{1+\varrho(\varsigma,\varpi)}\right)^{\lambda_4}.\left(\frac{\varrho(\varsigma,\Upsilon\varpi)(1+\varrho(\varpi,\Upsilon\varsigma))}{2}\right)^{\lambda_5} \\ \qquad\qquad \text{for } s = 0,\ \varsigma, \varpi \in \Xi \backslash fix(\Upsilon). \end{cases}$$

Here, $fix(\Upsilon) = \{\varsigma \in \Xi : \varsigma = \Upsilon\varsigma\}$

3. Main Results

We begin this section by defining the notion of admissible hybrid (θ-ζ)-contraction in the setting of GMS.

Definition 3.1. *Let* (Ξ, ϱ) *be a GMS. A mapping* $\Upsilon : \Xi \longrightarrow \Xi$ *is said to be an admissible hybrid (θ-ζ)-contraction if there exist* $\theta \in \Theta$, $\zeta \in \Phi$, $r \in (0,1)$ *and a mapping* $\alpha : \Xi \times \Xi \longrightarrow \mathbb{R}^+$ *such that*

$$\theta(\alpha(\varsigma, \varpi)\varrho(\Upsilon\varsigma, \Upsilon\varpi)) \leq [\theta(\zeta(M_{\lambda_i}(\varsigma, \varpi, s, \Upsilon)))]^r, \tag{3}$$

where $M_{\lambda_i}(\varsigma, \varpi, s, \Upsilon)$ *is as defined in* (2)

Theorem 3.2. *Let* (Ξ, ϱ) *be a complete GMS and* Υ *be an admissible hybrid (θ-ζ)-contraction. Suppose further that:*

(i) Υ *is triangular* α*-orbital admissible;*

(ii) there exists $\varsigma_0 \in \Xi$ *such that* $\alpha(\varsigma_0, \Upsilon\varsigma_0) \geq 1$*;*

(iii) either Υ *is continuous or;*

(iv) Υ^2 *is continuous and* $\alpha(\Upsilon\varsigma, \varsigma) \geq 1$ *for any* $\varsigma \in fix(\Upsilon)$.

Then, Υ *has a FP in* Ξ.

Proof. By hypothesis (ii), $\alpha(\varsigma_0, \Upsilon\varsigma_0) \geq 1$ for some $\varsigma_0 \in \Xi$. Define a sequence $\{\varsigma_n\}_{n\in\mathbb{N}}$ in Ξ by $\varsigma_n = \Upsilon^n\varsigma_0$, for all $n \in \mathbb{N}$. Suppose that we can find some $n_0 \in \mathbb{N}$ such that $\varsigma_{n_0} = \varsigma_{n_0+1} = \Upsilon\varsigma_{n_0}$. This implies that ς_{n_0} is a FP of Υ and hence, the proof is complete.

Assume on the contrary that $\varsigma_n \neq \varsigma_{n-1}$ for all $n \in \mathbb{N}$. Since $\alpha(\varsigma_0, \Upsilon\varsigma_0) \geq 1$ and Υ is triangular α-orbital admissible, then,

$$\alpha(\varsigma_{n-1}, \varsigma_n) \geq 1 \text{ for all } n \in \mathbb{N}. \tag{4}$$

Given the fact that Υ is an admissible hybrid (θ-ζ)-contraction, it follows that

$$\theta(\alpha((\varsigma_{n-1}, \varsigma_n))\varrho(\Upsilon\varsigma_{n-1}, \Upsilon\varsigma_n)) \leq [\theta(\zeta(M_{\lambda_i}((\varsigma_{n-1}, \varsigma_n), s, \Upsilon)))]^r. \tag{5}$$

Combining (4) and (5) yields

$$\begin{aligned}\theta(\varrho(\varsigma_n, \varsigma_{n+1})) &\leq \theta(\alpha(\varsigma_{n-1}, \varsigma_n)\varrho(\Upsilon\varsigma_{n-1}, \Upsilon\varsigma_n)) \\ &\leq [\theta(\zeta(M_{\lambda_i}(\varsigma_{n-1}, \varsigma_n, s, \Upsilon)))]^r.\end{aligned} \tag{6}$$

In furthering the arguments, the following cases are considered:

Case 1: for $s > 0$, let $\varsigma = \varsigma_{n-1}$ and $\varpi = \varsigma_n$. Then,

$$\begin{aligned} M_{\lambda_i}(\varsigma_{n-1}, \varsigma_n) =& \Bigg[\lambda_1 \varrho(\varsigma_{n-1}, \varsigma_n)^s + \lambda_2 \varrho(\varsigma_{n-1}, \Upsilon\varsigma_{n-1})^s + \lambda_3 \varrho(\varsigma_n, \Upsilon\varsigma_n)^s \\ &+ \lambda_4 \left(\frac{\varrho(\varsigma_n, \Upsilon\varsigma_n)(1+\varrho(\varsigma_{n-1}, \Upsilon\varsigma_{n-1}))}{1+\varrho(\varsigma_{n-1}, \varsigma_n)}\right)^s \\ &+ \lambda_5 \left(\frac{\varrho(\varsigma_n, \Upsilon\varsigma_{n-1})(1+\varrho(\varsigma_{n-1}, \Upsilon\varsigma_n))}{1+\varrho(\varsigma_{n-1}, \varsigma_n)}\right)^s\Bigg]^{\frac{1}{s}} \\ =& \Bigg[\lambda_1 \varrho(\varsigma_{n-1}, \varsigma_n)^s + \lambda_2 \varrho(\varsigma_{n-1}, \varsigma_n)^s + \lambda_3 \varrho(\varsigma_n, \varsigma_{n+1})^s \\ &+ \lambda_4 \left(\frac{\varrho(\varsigma_n, \varsigma_{n+1})(1+\varrho(\varsigma_{n-1}, \varsigma_n))}{1+\varrho(\varsigma_{n-1}, \varsigma_n)}\right)^s \\ &+ \lambda_5 \left(\frac{\varrho(\varsigma_n, \varsigma_n)(1+\varrho(\varsigma_{n-1}, \varsigma_{n+1}))}{1+\varrho(\varsigma_{n-1}, \varsigma_n)}\right)^s\Bigg]^{\frac{1}{s}} \\ =& \Bigg[\lambda_1 \varrho(\varsigma_{n-1}, \varsigma_n)^s + \lambda_2 \varrho(\varsigma_{n-1}, \varsigma_n)^s + \lambda_3 \varrho(\varsigma_n, \varsigma_{n+1})^s \\ &+ \lambda_4 (\varrho(\varsigma_n, \varsigma_{n+1}))^s\Bigg]^{\frac{1}{s}} \\ =& \Big[(\lambda_1+\lambda_2)\varrho(\varsigma_{n-1,\varsigma_n})^s + (\lambda_3+\lambda_4)\varrho(\varsigma_n, \varsigma_{n+1})^s\Big]^{\frac{1}{s}}. \end{aligned} \tag{7}$$

Suppose that

$$\varrho(\varsigma_{n-1}, \varsigma_n) \leq \varrho(\varsigma_n, \varsigma_{n+1}).$$

Then, from (6) and (7),

$$\begin{aligned} \theta(\varrho(\varsigma_n, \varsigma_{n+1})) \leq& [\theta(\zeta(M_{\lambda_i}(\varsigma_{n-1}, \varsigma_n, s, \Upsilon)))]^r \\ =& [\theta(\zeta(\lambda_1+\lambda_2)\varrho(\varsigma_{n-1,\varsigma_n})^s + (\lambda_3+\lambda_4)\varrho(\varsigma_n, \varsigma_{n+1})^s)^{\frac{1}{s}}]^r \\ \leq& [\theta(\zeta(\lambda_1+\lambda_2)\varrho(\varsigma_n, \varsigma_{n+1})^s + (\lambda_3+\lambda_4)\varrho(\varsigma_n, \varsigma_{n+1})^s)^{\frac{1}{s}}]^r \\ =& [\theta(\zeta(\lambda_1+\lambda_2+\lambda_3+\lambda_4)^{\frac{1}{s}}\varrho(\varsigma_n, \varsigma_{n+1}))]^r \\ \leq& [\theta(\zeta(\varrho(\varsigma_n, \varsigma_{n+1})))]^r \\ <& [\theta(\varrho(\varsigma_n, \varsigma_{n+1}))]^r. \end{aligned}$$

That is, $\theta(\varrho(\varsigma_n, \varsigma_{n+1})) < [\theta(\varrho(\varsigma_n, \varsigma_{n+1}))]^r$, which is a contradiction for all $r \in (0,1)$. Hence,

$$\begin{aligned} \theta(\varrho(\varsigma_n, \varsigma_{n+1})) &\leq [\theta(\zeta(\varrho(\varsigma_{n-1}, \varsigma_n)))]^r \\ &\leq [\theta(\zeta(\zeta(\varrho(\varsigma_{n-2}, \varsigma_{n-1}))))]^{r^2} \\ &= \theta(\zeta^2(\varrho(\varsigma_{n-2}, \varsigma_{n-1})))]^{r^2} \\ &\leq \\ &\vdots \\ &\leq [\theta(\zeta^n(\varrho(\varsigma_0, \varsigma_1)))]^{r^n} \text{ for all } n \in \mathbb{N}. \end{aligned} \tag{8}$$

Thus, we have

$$1 \leq \theta(\varrho(\varsigma_n, \varsigma_{n+1})) \leq [\theta(\zeta^n(\varrho(\varsigma_0, \varsigma_1))]^{r^n}. \tag{9}$$

Letting $n \longrightarrow \infty$ in (9) and using Sandwich theorem, yield:

$$\theta(\varrho(\varsigma_n, \varsigma_{n+1})) \to 1 \text{ as } n \longrightarrow \infty,$$

which implies from (Θ_2) that

$$\lim_{n\to\infty} \varrho(\varsigma_n, \varsigma_{n+1}) = 0.$$

From condition (Θ_3), there exist $r \in (0,1)$ and $\ell \in (0, \infty]$ such that

$$\lim_{n\to\infty} \frac{\theta(\varrho(\varsigma_n, \varsigma_{n+1})) - 1}{[\varrho(\varsigma_n, \varsigma_{n+1})]^r} = \ell.$$

Suppose that $\ell < \infty$. In this case, let $B = \frac{\ell}{2} > 0$. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that

$$\left| \frac{\theta(\varrho(\varsigma_n, \varsigma_{n+1})) - 1}{[\varrho(\varsigma_n, \varsigma_{n+1})]^r} - \ell \right| \leq B \text{ for all n} \geq \text{n}_0.$$

This implies that

$$\frac{\theta(\varrho(\varsigma_n, \varsigma_{n+1})) - 1}{(\varrho(\varsigma_n, \varsigma_{n+1}))^r} \geq \ell - B = B, \text{ for all } n \geq n_0.$$

Then,

$$n[\varrho(\varsigma_n, \varsigma_{n+1})]^r \leq An[\theta(\zeta^n(\varrho(\varsigma_n, \varsigma_{n+1}))) - 1], \text{ for all } n \geq n_0,$$

where $A = \frac{1}{B}$. Suppose now that $\ell = \infty$. Let $B > 0$ be an arbitrary positive number. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that

$$\frac{\theta(\varrho(\varsigma_n, \varsigma_{n+1})) - 1}{(\varrho(\varsigma_n, \varsigma_{n+1}))^r} \geq B, \text{ for all } n \geq n_0.$$

This implies that

$$n[\varrho(\varsigma_n, \varsigma_{n+1})]^r \leq An[\theta(\zeta^n(\varrho(\varsigma_n, \varsigma_{n+1}))) - 1], \text{ for all } n \geq n_0,$$

where $A = \frac{1}{B}$. Thus, in all cases, there exist $A > 0$ and $n_0 \in \mathbb{N}$ such that

$$n[\varrho(\varsigma_n, \varsigma_{n+1})]^r \leq An[\theta(\varrho(\varsigma_n, \varsigma_{n+1})) - 1], \text{ for all } n \geq n_0.$$

Using (9), we obtain

$$n[\varrho(\varsigma_n, \varsigma_{n+1})]^r \leq An([\theta(\zeta^n(\varrho(\varsigma_0, \varsigma_1))]^{r^n} - 1), \quad \text{for all } n \geq n_0.$$

By allowing $n \to \infty$ in the inequality above, we get

$$\lim_{n\to\infty} n[\varrho(\varsigma_n, \varsigma_{n+1})]^r = 0.$$

Consequently, $n_1 \in \mathbb{N}$ exists such that

$$\varrho(\varsigma_n, \varsigma_{n+1}) \leq \frac{1}{n^{\frac{1}{r}}}, \text{ for all } n \geq n_1. \tag{10}$$

Now, we shall prove that Υ has a periodic point. Suppose that it is not the case, then $\varsigma_n \neq \varsigma_m$ for every $n, m \in \mathbb{N}$ such that $n \neq m$. Using (8), we obtain

$$\begin{aligned} \theta(\varrho(\varsigma_n, \varsigma_{n+2})) &\leq [\theta(\zeta(\varrho(\varsigma_{n-1}, \varsigma_{n+1})))]^r \\ &\leq [\theta(\zeta^2(\varrho(\varsigma_{n-2}, \varsigma_n)))]^{r^2} \\ &\leq \\ &\vdots \\ &\leq [\theta(\zeta^n(\varrho(\varsigma_0, \varsigma_2)))]^{r^n}. \end{aligned}$$

Letting $n \longrightarrow \infty$ in the above inequality and using (Θ_2), we have

$$\lim_{n\to\infty} \varrho(\varsigma_n, \varsigma_{n+2}) = 0.$$

Similarly, from condition (Θ_3), there exists $n_2 \in \mathbb{N}$ such that

$$\varrho(\varsigma_n, \varsigma_{n+2}) \leq \frac{1}{n^{\frac{1}{r}}}, \text{ for all } n \geq n_2. \tag{11}$$

Let $H = \max\{n_0, n_1\}$. Then, two instances are considered as follows:

(A) If $m > 2$ is odd, then writing $m = 2l + 1$, $l \geq 1$ and using (10) , for all $n \geq H$, gives

$$\begin{aligned}\varrho(\varsigma_n, \varsigma_{n+m}) \leq \varrho(\varsigma_n, \varsigma_{n+1}) &+ \varrho(\varsigma_{n+1}, \varsigma_{n+2}) \\ &+ \cdots + \varrho(\varsigma_{n+2l}, \varsigma_{n+2l+1}) \\ &\leq \frac{1}{n^{\frac{1}{r}}} + \frac{1}{(n+1)^{\frac{1}{r}}} + \cdots + \frac{1}{(n+2l)^{\frac{1}{r}}} \\ &= \sum_{i=n}^{n+2l} \frac{1}{i^{\frac{1}{r}}} \\ &\leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{r}}}.\end{aligned}$$

(B) If $m > 2$ is even, then writing $m = 2l$, $l \geq 2$ and using (10) and (11), yields

$$\begin{aligned}\varrho(\varsigma_n, \varsigma_{n+m}) \leq \varrho(\varsigma_n, \varsigma_{n+2}) &+ \varrho(\varsigma_{n+2}, \varsigma_{n+3}) \\ &+ \cdots + \varrho(\varsigma_{n+2l-1}, \varsigma_{n+2l}) \\ &\leq \frac{1}{n^{\frac{1}{r}}} + \frac{1}{(n+2)^{\frac{1}{r}}} + \cdots + \frac{1}{(n+2l-1)^{\frac{1}{r}}} \\ &\leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{r}}}.\end{aligned}$$

Thus, combining all the instances, leads to

$$\varrho(\varsigma_n, \varsigma_{n+m}) \leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{r}}} \text{ for all } n \geq H,\ m \in \mathbb{N}.$$

The series $\sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{r}}}$ converges since $\frac{1}{r} > 1$, which shows that $\{\varsigma_n\}$ is a Cauchy sequence. Since (Ξ, ϱ) is complete, there exists $u \in \Xi$ such that $\varsigma_n \longrightarrow u$ as $n \longrightarrow \infty$. Using assumption (iii) that Υ is continuous, leads to

$$\varrho(u, \Upsilon u) = \lim_{n\to\infty} \varrho(\varsigma_n, \Upsilon\varsigma_n) = \lim_{n\to\infty} \varrho(\varsigma_n, \varsigma_{n+1}) = \varrho(u, u) = 0.$$

This implies that $u = \Upsilon u$. Again, from (iv) that Υ^2 is continuous, we have $\Upsilon^2 u = \lim_{n\to\infty} \Upsilon^2 \varsigma_n = u$. To see that $\Upsilon u = u$, suppose on the opposite that $\Upsilon u \neq u$, and using the idea in (6), produces

$$\begin{aligned}\theta(\varrho(u, \Upsilon u)) = \theta(\varrho(\Upsilon^2 u, \Upsilon u)) &\leq \theta(\alpha(\Upsilon u, u)\varrho(\Upsilon u, u)) \\ &\leq [\theta(\zeta(M_{\lambda_i}(\Upsilon u, u)))]^r \\ &< \theta(M_{\lambda_i}(\Upsilon u, u))]^r,\end{aligned} \tag{12}$$

where

$$\begin{aligned} M_{\lambda_i}(\Upsilon u, u) &= \Bigg[\lambda_1(\varrho(\Upsilon u, u))^s + \lambda_2(\varrho(\Upsilon u, \Upsilon^2 u))^s + \lambda_3(\varrho(u, \Upsilon u))^s \\ &+ \lambda_4\left(\frac{\varrho(u, \Upsilon u)(1+\varrho(\Upsilon u, \Upsilon^2 u))}{1+\varrho(\Upsilon u, u)}\right)^s \\ &+ \lambda_5\left(\frac{\varrho(u, \Upsilon^2 u)(1+\varrho(\Upsilon u, \Upsilon u))}{1+\varrho(\Upsilon u, u)}\right)^s\Bigg]^{\frac{1}{s}} \\ &= \Bigg[\lambda_1(\varrho(\Upsilon u, u))^s + \lambda_2(\varrho(\Upsilon u, u))^s + \lambda_3(\varrho(u, \Upsilon u)^s \\ &+ \lambda_4\left(\frac{\varrho(u, \Upsilon u)(1+\varrho(\Upsilon u, u))}{1+\varrho(\Upsilon u, u)}\right)^s\Bigg)^s\Bigg]^{\frac{1}{s}} \\ &= \Big[(\lambda_1+\lambda_2+\lambda_3+\lambda_4)\varrho(\Upsilon u, u)^s\Big]^{\frac{1}{s}} \\ &\leq \varrho(\Upsilon u, u). \end{aligned}$$

Hence, (12) becomes $\theta(\varrho(u, \Upsilon u)) < [\theta(\varrho(u, \Upsilon u))]^r$, which is a contradiction for all $r \in (0, 1)$. Therefore, $\Upsilon u = u$.
Case 2: for s=0, let $\varsigma = \varsigma_{n-1}$ and $\varpi = \varsigma_n$. Then,

$$\begin{aligned} M_{\lambda_i}(\varsigma_{n-1}, \varsigma_n) =&(\varrho(\varsigma_{n-1}, \varsigma_n))^{\lambda_1}.(\varrho(\varsigma_{n-1}, \Upsilon\varsigma_{n-1}))^{\lambda_2}.(\varrho(\varsigma_n, \Upsilon\varsigma_n))^{\lambda_3}. \\ &\left(\frac{\varrho(\varsigma_n, \Upsilon\varsigma_n)(1+\varrho(\varsigma_{n-1}, \Upsilon\varsigma_{n-1}))}{1+\varrho(\varsigma_{n-1}, \varsigma_n)}\right)^{\lambda_4}.\left(\frac{\varrho(\varsigma_{n-1}, \Upsilon\varsigma_n)(1+\varrho(\varsigma_n, \Upsilon\varsigma_{n-1}))}{2}\right)^{\lambda_5} \\ =&(\varrho(\varsigma_{n-1}, \varsigma_n))^{\lambda_1}.(\varrho(\varsigma_{n-1}, \Upsilon\varsigma_{n-1}))^{\lambda_2}.(\varrho(\varsigma_n, \varsigma_{n+}))^{\lambda_3}. \\ &\left(\frac{\varrho(\varsigma_n, \varsigma_{n+})(1+\varrho(\varsigma_{n-1}, \Upsilon\varsigma_{n-1}))}{1+\varrho(\varsigma_{n-1}, \varsigma_n)}\right)^{\lambda_4}.\left(\frac{\varrho(\varsigma_{n-1}, \varsigma_{n+1})(1+\varrho(\varsigma_n, \varsigma_n))}{2}\right)^{\lambda_5} \\ \leq&(\varrho(\varsigma_{n-1}, \varsigma_n))^{(\lambda_1+\lambda_2)}.(\varrho(\varsigma_n, \varsigma_{n+1}))^{(\lambda_3+\lambda_4)}.\left(\frac{\varrho(\varsigma_{n-1}, \varsigma_n)+\varrho(\varsigma_n, \varsigma_{n+1}))}{2}\right)^{\lambda_5} \\ \leq&(\varrho(\varsigma_{n-1}, \varsigma_n))^{(\lambda_1+\lambda_2)}.(\varrho(\varsigma_n, \varsigma_{n+1}))^{(\lambda_3+\lambda_4)}.\frac{(\varrho(\varsigma_{n-1}, \varsigma_n))^{\lambda_5}+(\varrho(\varsigma_n, \varsigma_{n+1}))^{\lambda_5}}{2}. \end{aligned}$$

Suppose that $\varrho(\varsigma_{n-1}, \varsigma_n) \leq \varrho(\varsigma_n, \varsigma_{n+1})$, then,

$$\begin{aligned} M_{\lambda_i}(\varsigma_n, \varsigma_{n+1}) \leq&(\varrho(\varsigma_{n-1}, \varsigma_n))^{(\lambda_1+\lambda_2)}.(\varrho(\varsigma_n, \varsigma_{n+1}))^{(\lambda_3+\lambda_4)}.(\varrho(\varsigma_n, \varsigma_{n+1}))^{\lambda_5} \\ =&(\varrho(\varsigma_n, \varsigma_{n+1}))^{(\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5)} \\ =&\varrho(\varsigma_n, \varsigma_{n+1}). \end{aligned}$$

Hence, (6) can be written as

$$\begin{aligned} \theta(\varrho(\varsigma_n, \varsigma_{n+1})) \leq&[\theta(\zeta(\varrho(\varsigma_{n-1}, \varsigma_n)))]^r \\ <&[\theta(\varrho(\varsigma_n, \varsigma_{n+1}))]^r, \end{aligned}$$

which is a contradiction for all $r \in (0,1)$. Therefore, by (6),

$$\begin{aligned}\theta(\varrho(\varsigma_n,\varsigma_{n+1})) &\leq [\theta(\zeta(\varrho(\varsigma_{n-1},\varsigma_n)))]^r\\ &\leq [\theta(\zeta(\zeta(\varrho(\varsigma_{n-2},\varsigma_{n-1}))))]^{r^2}\\ &= [\theta(\zeta^2(\varrho(\varsigma_{n-2},\varsigma_{n-1})))]^{r^2}\\ &\leq\\ &\vdots\\ &\leq [\theta(\zeta^n(\varrho(\varsigma_0,\varsigma_1))))]^{r^n}.\end{aligned}$$

Using the same argument as the case of $s>0$, it can be obtained that $\{\varsigma_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Since (Ξ,ϱ) is complete, so there exists a point say u such that $\lim_{n\to\infty}\varsigma_n = u$. to see that u is a FP of Υ, from assumption (iii), we have

$$\varrho(u,\Upsilon u) = \lim_{n\to\infty}\varrho(\varsigma_n,\Upsilon\varsigma_n) = \lim_{n\to\infty}\varrho(\varsigma_n,\varsigma_{n+1}) = \varrho(u,u) = 0.$$

This implies that $u = \Upsilon u$. Again, from (iv) under the assumption that Υ^2 is continuous as in case (i), we have

$$\begin{aligned}\theta(\varrho(u,\Upsilon u)) = \theta(\varrho(\Upsilon^2 u,\Upsilon u)) &\leq \theta(\alpha(\Upsilon u,u)\varrho(\Upsilon u,u))\\ &\leq [\theta(\zeta(M_{\lambda_i}(\Upsilon u,u)))]^r\\ &< \theta(M_{\lambda_i}(\Upsilon u,u))]^r\\ &= [\theta(\varrho(u,\Upsilon u))^{\lambda_1+\lambda_2+\lambda_3+\lambda_+4+\lambda_5}]^r\\ &= [\theta(\varrho(u,\Upsilon u))]^r.\end{aligned}$$

That is, $\theta(\varrho(u,\Upsilon u)) < [\theta(\varrho(u,\Upsilon u))]^r$, which is a contradiction for all $r\in(0,1)$. It follows that $u=\Upsilon u$. □

Theorem 3.3. *If in Theorem 3.2, we assume an additional condition that $\alpha(\varsigma,\varpi)\geq 1$ for all $\varsigma,\varpi\in fix(\Upsilon)$. Then, the FP of Υ is unique.*

Proof. Let ς, u be two FP of Υ such that $\Upsilon\varsigma = \varsigma \neq u = \Upsilon u$, then $\varrho(\varsigma,u) = \varrho(\Upsilon\varsigma,\Upsilon u)\neq 0$ and using (6) implies

$$\begin{aligned}\theta(\varrho(\varsigma,u)) = \theta(\varrho(\Upsilon\varsigma,\Upsilon u)) &\leq \theta(\alpha(\varsigma,u)\varrho(\Upsilon\varsigma,\Upsilon\varpi))\\ &\leq [\theta(\zeta(M_{\lambda_i}(\varsigma,u,s,\Upsilon)))]^r.\end{aligned} \tag{13}$$

Case 1: for $s>0$

$$\begin{aligned}M_{\lambda_i}(\varsigma,u) &= \Bigg[\lambda_1(\varrho(\varsigma,u))^s + \lambda_2(\varrho(\varsigma,\Upsilon\varsigma))^s + \lambda_3(\varrho(u,\Upsilon u))^s\\ &\quad + \lambda_4\left(\frac{\varrho(u,\Upsilon u)(1+\varrho(\varsigma,\Upsilon\varsigma))}{1+\varrho(\varsigma,u)}\right)^s\\ &\quad + \lambda_5\left(\frac{\varrho(u,\Upsilon\varsigma)(1+\varrho(\varsigma,\Upsilon u))}{1+\varrho(\varsigma,u)}\right)^s\Bigg]^{\frac{1}{s}}\\ &= \Bigg[\lambda_1(\varrho(\varsigma,u))^s + \lambda_2(\varrho(\varsigma,\varsigma))^s + \lambda_3(\varrho(u,u))^s\\ &\quad + \lambda_4\left(\frac{\varrho(u,u)(1+\varrho(\varsigma,\varsigma))}{1+\varrho(\varsigma,u)}\right)^s + \lambda_5\left(\frac{\varrho(u,\varsigma)(1+\varrho(\varsigma,u))}{1+\varrho(\varsigma,u)}\right)^s\Bigg]^{\frac{1}{s}}\\ &= \Big[\lambda_1(\varrho(\varsigma,u))^s + \lambda_5(\varrho(\varsigma,u))^s\Big]^{\frac{1}{s}}\\ &= (\lambda_1+\lambda_5)^{\frac{1}{s}}\varrho(\varsigma,u) \leq \varrho(\varsigma,u).\end{aligned}$$

Hence, (13) can be written as

$$\theta(\varrho(\varsigma,u)) \leq [\theta(\zeta((\varrho(\varsigma,u))))]^r < [\theta(\varrho(\varsigma,u))]^r,$$

which is a contradiction for all $r \in (0,1)$. Thus, $\varsigma = u$.
case 2: for $s = 0$

$$\begin{aligned} M_{\lambda_i}(\varsigma,u) =&(\varrho(\varsigma,u))^{\lambda_1}.(\varrho(\varsigma,\Upsilon\varsigma))^{\lambda_2}.(\varrho(u,\Upsilon u))^{\lambda_3}. \\ &\left(\frac{\varrho(u,\Upsilon u)(1+\varrho(\varsigma,\Upsilon\varsigma))}{1+\varrho(\varsigma,u)}\right)^{\lambda_4}.\left(\frac{\varrho(\varsigma,\Upsilon u)(1+\varrho(u,\Upsilon\varsigma))}{2}\right)^{\lambda_5} \\ =&(\varrho(\varsigma,u))^{\lambda_1}.(\varrho(\varsigma,\varsigma))^{\lambda_2}.(\varrho(u,u))^{\lambda_3}. \\ &\left(\frac{\varrho(u,u)(1+\varrho(\varsigma,\varsigma))}{1+\varrho(\varsigma,u)}\right)^{\lambda_4}.\left(\frac{\varrho(\varsigma,u)(1+\varrho(u,\varsigma))}{2}\right)^{\lambda_5} \\ =&0. \end{aligned}$$

Therefore, (13) becomes

$$\theta(\varrho(\varsigma,u)) \leq [\theta(\zeta((0)))]^r < [\theta(0)]^r,$$

which is a contradiction for all $r \in (0,1)$. Hence $\varsigma = u$, showing that the FP of Υ is unique. □

Corollary 3.4. *Let (Ξ,ϱ) be a CMS and $\Upsilon : \Xi \longrightarrow \Xi$ be a given mapping. Suppose that there exists $r \in (0,1)$ such that*

$$\theta(\varrho(\Upsilon\varsigma,\Upsilon\varpi)) \leq [\theta(\zeta(M_{\lambda_i}(\varsigma,\varpi,s,\Upsilon)))]^r$$

for all $\varsigma,\varpi \in \Xi\backslash fix(\Upsilon)$, where θ satisfies $(\Theta_1)-(\Theta_3)$, $\zeta \in \Phi$ and $M_{\lambda_i}(\varsigma,\varpi,s,\Upsilon)$ is as given in 2. Then, Υ has a unique FP in Ξ.

Proof. The proof is immediate from theorem 3.2 by taking $\alpha(\varsigma,\varpi) = 1$. □

Corollary 3.5. *Given that (Ξ,ϱ) is a CMS and $\Upsilon : \Xi \longrightarrow \Xi$ be a given mapping fulfilling:*

$$\varrho(\Upsilon\varsigma,\Upsilon\varpi) \leq \eta(M_{\lambda_i}(\varsigma,\varpi,s,\Upsilon)),$$

for all $\varsigma,\varpi \in \Xi\backslash fix(\Upsilon)$ and $\eta \in (0,1)$. then, Υ possesses a unique FP in Ξ.

Proof. From Corollary 3.4, let $\theta(t) = e^t$ and $\zeta(t) = \eta t$; $t > 0$, then the conclusion is fulfilled. □

Corollary 3.6. *[10] Let (Ξ,ϱ) be a CMS and $\Upsilon : \Xi \longrightarrow \Xi$ be a mapping fulfilling:*

$$\theta(\varrho(\Upsilon\varsigma,\Upsilon^2\varsigma)) \leq [\theta(\varrho(\varsigma,\Upsilon\varsigma))]^k, \tag{14}$$

for all $\varsigma \in \Xi$ with $\varrho(\Upsilon\varsigma,\Upsilon^2\varsigma) > 0$, where θ is non-decreasing, continuous and satisfies Θ_2. Then, Υ has a unique FP in Ξ.

Note that a self mapping Υ is said to have the property P if $Fix(\Upsilon^n) = Fix(\Upsilon)$ for all $n \in \mathbb{N}$. Note that Υ^n is the n-th iterate of the mapping Υ.

Corollary 3.7. *[22] Let (Ξ,ϱ) be a CMS $\Upsilon : \Xi \longrightarrow \Xi$ be a given mapping. If there exist $r \in (0,1)$ and $\theta \in \Theta_2$ such that for all $\varsigma,\varpi \in \Xi$,*

$$\varrho(\Upsilon\varsigma,\Upsilon\varpi) > 0 \Rightarrow \theta(\varrho(\Upsilon\varsigma,\Upsilon\varpi)) \leq [\theta(M(\varsigma,\varpi))]^k,$$

where

$$M(\varsigma,\varpi) = \max\{\varrho(\varsigma,\varpi),\varrho(\varsigma,\Upsilon\varsigma),\varrho(\varpi,\Upsilon\varpi)\}.$$

Then, the FP of Υ is unique in Ξ.

Proof. Consider the case when $s > 0$ in Corollary 3.4 and take $\lambda_4 = \lambda_5 = 0$. Then the proof is immediate. □

Example 3.8. *Let* $\Xi = \{\varsigma_n = \frac{n(3n+1)}{2}: n \in \mathbb{N}\} \bigcup \{\frac{n}{n+1}\}_{n\geq 1}$. *Take* $\varrho(\varsigma, \varpi) = |\varsigma - \varpi|$ *for all* $\varsigma, \varpi \in \Xi$, *then* (Ξ, ϱ) *is a CMS. Consider the mapping* $\Upsilon : \Xi \longrightarrow \Xi$ *defined by*

$$\Upsilon\varsigma_1 = \varsigma_1 \text{ and } \Upsilon\varsigma_n = \varsigma_{n-1} \text{ if } n \geq 2,$$

and define the mapping $\alpha : \Xi \times \Xi \longrightarrow \mathbb{R}_+$ *by*

$$\alpha(\varsigma_n, \varsigma_m) = \begin{cases} 1, & \textit{if } n, m \in \mathbb{N}; \\ 0, & \textit{otherwise}. \end{cases}$$

Observe that the Banach contraction is not fulfilled. In fact, we can check easily that

$$\lim_{n\to\infty} \frac{\varrho(\Upsilon\varsigma_n, \Upsilon\varsigma_1)}{\varrho(\varsigma_n, \varsigma_1)} = \lim_{n\to\infty} \frac{3n^2 - 5n - 2}{3n^2 + n - 4} = 1.$$

Note also that for all $\varsigma, m \notin \mathbb{N}$, *there is nothing to show. To see that* Υ *is an admissible hybrid (θ-ζ)-contraction, it is sufficient to show that*

$$\theta(\alpha(\varsigma_n, \varsigma_m)\varrho(\Upsilon\varsigma_n, \Upsilon\varsigma_m)) \leq [\theta(\zeta(M_{\lambda_i}(\varsigma_n, \varsigma_m, s, \Upsilon)))]^r, \tag{15}$$

holds for some $r \in (0, 1)$ *and for all* $n, m \in \mathbb{N}$.
Now, consider the function $\theta \in \Theta$ *defined by* $\theta(t) = \sqrt{te^t}$; $t > 0$, *then for all* $n, m \in \mathbb{N}$, (15) *becomes*

$$\begin{aligned} \varrho(\Upsilon\varsigma_n, \Upsilon\varsigma_m)e^{\varrho(\Upsilon\varsigma_n,\Upsilon\varsigma_m)} &\leq r^2[\zeta(M_{\lambda_i}(\varsigma_n, \varsigma_m, s, \Upsilon))e^{\zeta(M_{\lambda_i}(\varsigma_n,\varsigma_m,s,\Upsilon))}] \\ &< r^2(M_{\lambda_i}(\varsigma_n, \varsigma_m, s, \Upsilon)e^{M_{\lambda_i}(\varsigma_n,\varsigma_m,s,\Upsilon)}. \end{aligned} \tag{16}$$

It can be seen that for the both cases where $s > 0$ *and* $s = 0$, *let* $\lambda_1 = 1$ *and* $\lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0$, *then* $M_{\lambda_i}(\varsigma_n, \varsigma_m, s, \Upsilon) = \varrho(\varsigma_n, \varsigma_m)$. *This implies that* (16) *can be written as*

$$\varrho(\Upsilon\varsigma_n, \Upsilon\varsigma_m)e^{\varrho(\Upsilon\varsigma_n,\Upsilon\varsigma_m)} \leq r^2\varrho(\varsigma_n, \varsigma_m)e^{\varrho(\varsigma_n,\varsigma_m)}. \tag{17}$$

Two instances are considered as follows:

(A) $n = 1$ *and* $m \geq 2$. *From* (17), *we have*

$$\begin{aligned} &\frac{\varrho(\Upsilon\varsigma_1, \Upsilon\varsigma_m)e^{\varrho(\Upsilon\varsigma_1,\Upsilon\varsigma_m)-\varrho(\varsigma_1,\varsigma_m)}}{\varrho(\varsigma_1, \varsigma_m)} \\ &= \frac{3m^2 - 5m - 2}{3m^2 + m - 4}e^{-(3m-1)} \\ &\leq e^{-1}. \end{aligned}$$

(B) $m > n > 1$. *From* (17), *we have*

$$\begin{aligned} &\frac{\varrho(\Upsilon\varsigma_n, \Upsilon\varsigma_m)e^{\varrho(\Upsilon\varsigma_n,\Upsilon\varsigma_m)-\varrho(\varsigma_n,\varsigma_m)}}{\varrho(\varsigma_n, \varsigma_m)} \\ &\frac{(3m + 3m - 5)(n - m)}{(3m + 3m + 1)(n - m)}e^{-3(m-n)} \\ &\leq e^{-1}. \end{aligned}$$

Thus, the inequality (17) is fulfilled with $r = e^{-\frac{1}{2}}$. This implies that Υ has a unique FP in Ξ. In this example, ς_1 is the unique fixed of Υ.
However, Υ is not an admissible hybrid contraction in the sense of Karapinar and Fulga [17]. To see this, define a mapping $\zeta \in \Phi$ by $\zeta(t) = \frac{t}{5}$ for all $t \geq 0$. Then, from 15, we have

$$\begin{aligned}\varrho(\Upsilon\varsigma_n, \Upsilon\varsigma_m)e^{\varrho(\Upsilon\varsigma_n,\Upsilon\varsigma_m)} &\leq r^2\zeta(M_{\lambda_i}(\varsigma_n, \varsigma_m, s, \Upsilon))e^{\zeta(M_{\lambda_i}(\varsigma_n,\varsigma_m,s,\Upsilon))}\\ &= r^2\frac{1}{5}(M_{\lambda_i}(\varsigma_n, \varsigma_m, s, \Upsilon))e^{\frac{1}{5}(M_{\lambda_i}(\varsigma_n,\varsigma_m,s,\Upsilon)}\\ &< r^2M_{\lambda_i}(\varsigma_n, \varsigma_m, s, \Upsilon)e^{M_{\lambda_i}(\varsigma_n,\varsigma_m,s,\Upsilon)}. \end{aligned} \tag{18}$$

Obviously, (17) and (18) coincide. Here, under the value of $\lambda_1 = 1$ and $\lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0$, it follows that the mapping Υ is an admissible hybrid (θ-ζ)-contraction. On the other hand, for all $\varsigma, \varpi \in \Xi \setminus fix(\Upsilon)$, (2) yields

$$\alpha(\varsigma_n, \varsigma_m)\varrho(\Upsilon\varsigma_n, \Upsilon\varsigma_m) \leq \zeta(\varrho(\varsigma_n, \varsigma_m)),$$

which implies that

$$\varrho(\Upsilon\varsigma_n, \Upsilon\varsigma_m) \leq \frac{1}{5}(\varrho(\varsigma_n, \varsigma_m)). \tag{19}$$

Pick two points $\varsigma_n = \varsigma_1$ and $\varsigma_m = \varsigma_5$. Then, (19) gives

$$\begin{aligned}\frac{5\varrho(\Upsilon\varsigma_1, \Upsilon\varsigma_5)}{\varrho(\varsigma_1, \varsigma_5)} = \frac{5\varrho(2, 26)}{\varrho(2, 40)} &= \frac{5|2 - 26|}{|2 - 40|}\\ &= \frac{5(24)}{38} > 1,\end{aligned}$$

which shows that 2 fails.

4. Applications to Nonlinear Volterra-Fredholm Integral Equation

In this section, one of the obtained results is used to investigate conditions for the existence and uniqueness of a solution to a nonlinear Volterra-Fredholm equation. To this effect, consider the nonlinear Volterra-Fredholm integral equation of the first kind given as

$$\varsigma(t) = \tau_1\int_0^t k_1(t, s, \varsigma(s))ds + \tau_2\int_0^h k_2(t, s, \varsigma(s))ds; \quad t \in [0, h], \tag{20}$$

where $\varsigma(t)$ is the unknown solution, $k_i(t, s, \varsigma(s))$ are smooth functions, τ_i, h are constants: $(i = 1, 2)$. Let $\Xi = C([0, h])$ be the set of all continuous real valued functions defined on Ξ with the supremum norm. If Ξ is equipped with the metric $\varrho : \Xi \times \Xi \longrightarrow \mathbb{R}_+$ defined by $\varrho(\varsigma, \varpi) = \sup\{|\varsigma(t) - \varpi(t)|, t \in [0, h]\}$ then, (Ξ, ϱ) is a CMS. Define a mapping $\Upsilon : \Xi \longrightarrow \Xi$ by

$$\Upsilon\varsigma(t) = \tau_1\int_0^t k_1(t, s, \varsigma(s))ds + \tau_2\int_0^h k_2(t, s, \varsigma(s))ds. \tag{21}$$

Then, z is a FP of Υ if and only if z is a solution to (20). Under the following assumptions, we now examine the solvability conditions of the nonlinear Volterra-Fredholm integration equation (20).

Theorem 4.1. *Suppose that the following conditions are fulfilled:*

(i) Υ is a continuous mapping and $k_i : [0, h] \times [0, h] \times \mathbb{R} \longrightarrow \mathbb{R}$;

(ii) for some constants A_i, there exist $r \in (0, 1)$ and $\zeta \in \Phi$ such that $|k_i(t, s, \varsigma(t)) - k_i(t, s, \varpi(\varsigma))| \leq A_i[\zeta|\varsigma(s) - \varpi(s)|]$, where $i = 1, 2$;

(iii) $\delta t + \eta h < 1 = r \in (0,1)$, *where* $\tau_1 A_1 = \delta$, *and* $\tau_2 A_2 = \eta$.

Then, the integral equation (20) *has a unique solution in* Ξ.

Proof. Observe that for all $\varsigma, \varpi \in \Xi$, using (21) and the hypotheses in Theorem 4.1,

$$\begin{aligned}
|\Upsilon\varsigma(t) - \Upsilon\varpi(t)| =& \left|\tau_1 \int_0^t k_1(t,s,\varsigma(s))ds + \tau_2 \int_0^h k_2(t,s,\varsigma(s))ds - \left[\tau_1 \int_0^t k_1(t,s,\varpi(s))ds \right.\right.\\
&\left.\left. + \tau_2 \int_0^h k_2(t,s,\varpi(s))ds\right]\right| \\
=& \left|\tau_1\left[\int_0^t k_1(t,s,\varsigma(s))ds - k_1(t,s,\varpi(s))ds\right] + \tau_2\left[\int_0^h k_2(t,s,\varsigma(s))ds - k_2(t,s,\varpi(s))ds\right]\right| \\
\leq& \tau_1 \int_0^t A_1\left[\zeta|\varsigma(s) - \varpi(s)|\right]ds + \tau_2 \int_0^h A_2\left[\zeta|\varsigma(s) - \varpi(s)|\right]ds \\
=& \tau_1 A_1\left[\zeta|\varsigma(s) - \varpi(s)|\right] \int_0^t ds + \tau_2 A_2\left[r\mu|\varsigma(s) - \varpi(s)|\right] \int_0^h ds \\
\leq& \tau_1 A_1[\zeta\|\varsigma - \varpi\|]t + \tau_2 A_2[\zeta\|\varsigma - \varpi\|]h \\
=& (\tau_1 A_1 t + \tau_2 A_2 h)[\zeta\|\varsigma - \varpi\|] \\
=& (\delta t + \eta h)[\zeta\|\varsigma - \varpi\|] \leq r\zeta\|\varsigma - \varpi\|.
\end{aligned}$$

This implies that $\sup_{t\in[0,h]} |\Upsilon\varsigma(t) - \Upsilon\varpi(t)| \leq r\zeta(\varrho(\varsigma,\varpi))$. Hence,

$$\varrho(\Upsilon\varsigma, \Upsilon\varpi) \leq r\zeta(\varrho(\varsigma,\varpi)). \tag{22}$$

Taking exponential of both sides of (22) and define a mapping $\theta \in \Theta$ by $\theta(t) = e^t$, then (22) yields

$$\theta(\varrho(\Upsilon\varsigma, \Upsilon\varpi)) \leq [\theta(\zeta(\varrho(\varsigma,\varpi)))]^r.$$

And it follows from Corollary 3.4 that Υ has a unique FP in Ξ by taking $\lambda_1 = 1$ and $\lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0$ for both cases when $s > 0$ and $s = 0$. □

5. Numerical Example

This section establishes a numerical example to support the reliability of the given results, using a class of integral equation. The ideas used in this section are motivated by [1]. Let $\Xi = C([0,1],\mathbb{R})$ be the set of all continuous real-valued functions, and $\Upsilon : \Xi \longrightarrow \Xi$ be a mapping defined by

$$\Upsilon\varsigma(t) = \psi(t) + \int_0^1 k(t,s)\varsigma(s)ds; \quad t,s \in [0,1]^2. \tag{23}$$

Let $\psi(t) = 2t\cos(t)$ and $k(t,s)\varsigma(s) = t^2\sin(\varsigma(s))$. Then, by substitution, equation (23) becomes

$$\Upsilon\varsigma(t) = 2t\cos(t) + \int_0^1 t^2\sin(\varsigma(s))ds. \tag{24}$$

Let $\zeta : [0,\infty) \longrightarrow [0,\infty)$ be a mapping fulfilling the condition:

(A) $|\sin(\varsigma(s)) - \sin(\varpi(s))| \leq r(\zeta|\varsigma(s) - \varpi(s)|)$ for some $r \in (0,1)$.

Note that for any $\varsigma, \varpi \in \Xi$ and using condition (A) in (24), we have

$$\begin{aligned}
\left|\Upsilon\varsigma(t) - \Upsilon\varpi(t)\right| =& \left|2t\cos(t) + \int_0^1 t^2 \sin(\varsigma(t))ds - 2t\cos(t) - \int_0^1 t^2 \sin(\varpi(t))ds\right| \\
=& \left|\int_0^1 t^2 \sin(\varsigma(t))ds - \int_0^1 t^2 \sin(\varpi(t))ds\right| \\
=& \left|\int_0^1 t^2 [\sin(\varsigma(t)) - \sin(\varpi(t))]ds\right| \\
\leq& \int_0^1 \left|t^2\right| \left[\left|\sin(\varsigma(t)) - \sin(\varpi(t))\right|\right] ds \\
\leq& \int_0^1 r(\zeta|\varsigma(t) - \varpi(t)|)ds \\
=& r(\zeta|\varsigma(s) - \varpi(s)|) \int_0^1 ds \\
=& r(\zeta|\varsigma(s) - \varpi(s)|).
\end{aligned}$$

This implies that

$$\varrho(\Upsilon\varsigma, \Upsilon\varpi) \leq r\zeta(\varrho(\varsigma, \varpi)). \tag{25}$$

Now, taking exponential of both sides of (25) and define a mapping $\theta \in \Theta$ by $\theta(t) = e^t$, then (25) yield

$$\theta(\varrho(\Upsilon\varsigma, \Upsilon\varpi)) \leq [\theta(\zeta(\varrho(\varsigma, \varpi)))]^r. \tag{26}$$

And it follows from Corollary 3.4 that Υ has a unique FP in Ξ by taking $\lambda_1 = 1$ and $\lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0$ for both cases when $s > 0$ and $s = 0$.

Furthermore, the iterative method will be employed to examine the viability of our technique. Tables 1,2, 3, 4 and 5 show the sequence of iteration of $\varsigma_{n+1}(t) = \Upsilon\varsigma_n(t) = 2t\cos(t) + \int_0^1 t^2 \sin(\varsigma_n(s))ds$. Let $\varsigma_0(t) = 0$ be an initial fixed solution.

Table 1: for $t = 0.2$

n	$\varsigma_{n+1}(0.2)$	Approximate Solution	Absolute Error
0	$\varsigma_1(0.2)$	0.399998	2×10^{-6}
1	$\varsigma_2(0.2)$	0.400277	2.77×10^{-4}
2	$\varsigma_3(0.2)$	0.400277	2.77×10^{-4}
3	$\varsigma_4(0.2)$	0.400277	2.77×10^{-4}

Table 2: for $t = 0.4$

n	$\varsigma_{n+1}(0.4)$	Approximate Solution	Absolute Error
0	$\varsigma_1(0.4)$	0.799981	1.9×10^{-5}
1	$\varsigma_2(0.4)$	0.802215	2.215×10^{-3}
2	$\varsigma_3(0.4)$	0.802221	2.221×10^{-3}
3	$\varsigma_4(0.4)$	0.802221	2.221×10^{-3}

Table 3: for $t = 0.6$

n	$\varsigma_{n+1}(0.6)$	Approximate Solution	Absolute Error
0	$\varsigma_1(0.6)$	1.1999342	6.6×10^{-5}
1	$\varsigma_2(0.6)$	1.207473	7.473×10^{-3}
2	$\varsigma_3(0.6)$	1.207520	7.52×10^{-3}
3	$\varsigma_4(0.6)$	1.207520	7.52×10^{-3}

Table 4: for $t = 0.8$

n	$\varsigma_{n+1}(0.8)$	Approximate Solution	Absolute Error
0	$\varsigma_1(0.8)$	1.599844	1.56×10^{-4}
1	$\varsigma_2(0.8)$	1.623296	2.3296×10^{-2}
2	$\varsigma_3(0.8)$	1.623640	2.364×10^{-2}
3	$\varsigma_4(0.8)$	1.623645	2.3645×10^{-2}
4	$\varsigma_5(0.8)$	1.623645	2.3645×10^{-2}

Table 5: for $t = 1$

n	$\varsigma_{n+1}(1)$	Approximate Solution	Absolute Error
0	$\varsigma_1(1)$	1.999695	3.05×10^{-4}
1	$\varsigma_2(1)$	2,034589	3.4589×10^{-2}
2	$\varsigma_3(1)$	2.035198	3.5198×10^{-2}
3	$\varsigma_4(1)$	2.035208	3.5208×10^{-2}
3	$\varsigma_4(1)$	2.035208	3.5208×10^{-2}

Note from the tables that $\varsigma(t) = 2t$ is the exact solution of equation (23). Moreover, Figures 1 and 2 show the convergence behaviours of the sequence $\varsigma_{n+1}(t) = \Upsilon\varsigma_n(t)$.

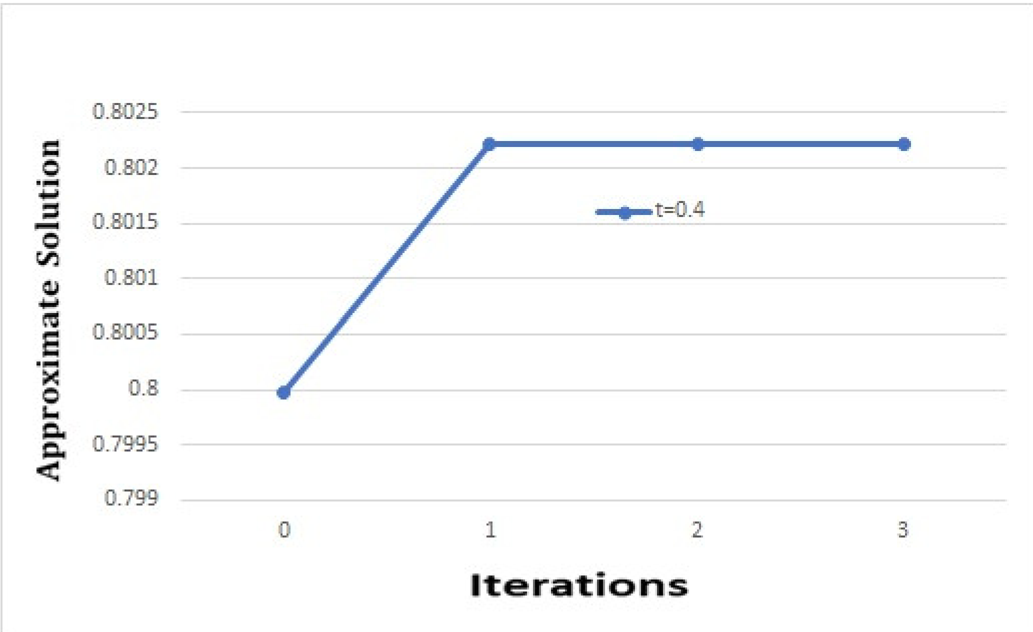

Figure 1: The graph shows that the sequence $\varsigma_{n+1}(t) = \Upsilon\varsigma_n(t) = 2t\cos(t) + \int_0^1 t^2 \sin(\varsigma_n(s))ds$ converges to the exact solution of $2t$ for $t = 0.4$.

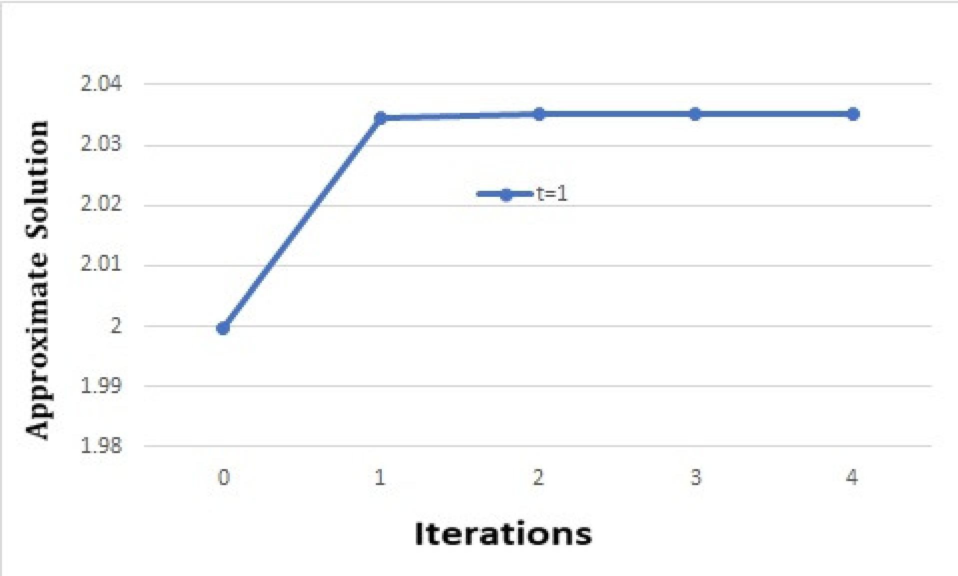

Figure 2: The graph shows that the sequence $\varsigma_{n+1}(t) = \Upsilon\varsigma_n(t) = 2t\cos(t) + \int_0^1 t^2 \sin(\varsigma_n(s))ds$ converges to the exact solution of $2t$ for $t = 1$.

6. Conclusion

This paper studied a new idea under the name admissible hybrid (θ-ζ)-contraction and examined conditions under which such mappings possess some FPs in the context of CMS. It has been deduced, by way of corollaries that the ideas proposed herein are new and improve some previously published results in the related literature. Regarding application, the existence and uniqueness of solutions to a mixed integral equations were established via the aid of one of the deduced consequences. Numerical example has also been constructed to further illustrate the effectiveness of the results obtained herein.

Data Availability

There is no data availability statement to be declared.

Conflicts of Interests

The authors declare that they have no competing interests.

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