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Periodic Solutions of Certain Higher Order Autonomous Differential Equations via Topological Degree Theory

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Abstract

In this paper, we give some sufficient conditions for the existence of periodic solutions for some autonomous nonlinear ordinary differential equations of order n. The proposed method is based on the use of Brouwer's degree and especially the homotopy invariant of the topological degree.

Keywords: Periodic solution; Autonomous Differential Equation; Homotopy invariant; Topological degree.
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1. Introduction

In the qualitative theory of differential equations the main problems are the study of their periodic solutions, their existence, their numbers and their stability. It is interesting to note that the existence of periodic solutions of nonlinear autonomous differential equations has not been extensively investigated. The Poincare-Bendixon theorem [19], which is a powerful tool for the investigation of periodic solutions of systems of second order differential equations, is not applicable for third and higher order systems. In what follows, we use the idea of Brower's degree theory (see, e.g. [10, 11, 13, 15, 16]) to prove the existence of periodic solutions of higher order systems.

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The existence of nontrivial periodic solution of autonomous nonlinear third, fourth and fifth order differential equations of the forms

$$x''' + f(x, x', x'') = 0, \qquad f(x, -x', x'') = -f(x, x', x''),$$

$$x^{(4)} + f(x, x', x'', x''') = 0, \qquad f(-x, x', -x'', x''') = -f(x, x', x'', x'''), \tag{1}$$

$$x^{(5)} + f(x, x', x'', x''', x^{(4)}) = 0, \quad f(x, -x', x'', -x''', x^{(4)}) = -f(x, x', x'', x''', x^{(4)}), \tag{2}$$

has been investigated in [4, 5, 17]. For periodic problems of ordinary differential equations of higher order, other results are found in [1, 2, 3, 8, 9, 14].

Let U be an open and bounded set in $\mathbb{R}^n$. For any $y \in \mathbb{R}^n$, we put

$$\mathcal{C}^r(\overline{U}, \mathbb{R}^n) = \Big\{ f \in \mathcal{C}^r(U, \mathbb{R}^n) : d^j f \in \mathcal{C}(\overline{U}, \mathbb{R}^n), 0 \leqslant j \leqslant r \Big\},$$

$$\mathcal{D}_y^r(\overline{U}, \mathbb{R}^n) = \Big\{ f \in \mathcal{C}^r(\overline{U}, \mathbb{R}^n) : y \notin f(\partial U) \Big\}, \qquad \mathcal{D}_y(\overline{U}, \mathbb{R}^n) = \mathcal{D}_y^0(\overline{U}, \mathbb{R}^n).$$

For $f \in \mathcal{C}^1(U, \mathbb{R}^n)$, the Jacobian of f in $x \in U$ is $J_f(x) = \det\big[\frac{\partial f_i}{\partial x_j}(x)\big]_{1 \le i,j \le n}$. We say y is a regular point of f if $J_f(x) \neq 0$ for all $x \in f^{-1}(y)$. Now, we define the topological degree.

Definition 1.1. *Suppose $f \in \mathcal{D}_y^1(\overline{U}, \mathbb{R}^n)$ and y is a regular value of f. Define the degree of f at y relative to U, as the integer*

$$\deg(f, U, y) = \sum_{x \in f^{-1}(y)} \operatorname{sgn}\big(J_f(x)\big). \tag{3}$$

According to the Sard's Lemma [15], the set of $f^{-1}(y)$ is finite. Also, the topological degree theory gives us information on the existence, number and nature of solutions of the equation $f(x) = y$. According to the Kronecker's existence theorem [15], if $\deg(f, U, y) \neq 0$ then $f(x) = y$ has at least one solution in U.

Theorem 1.2 (Homotopy Invariance Property, [11]). *If $f, g \in \mathcal{D}_y(\overline{U}, \mathbb{R}^n)$ and $H(t) = (1-t)f + tg \in \mathcal{D}_y(\overline{U}, \mathbb{R}^n)$ for all $t \in [0, 1]$, then*

$$\deg(f, U, y) = \deg(g, U, y).$$

Theorem 1.3 (Rouché Property, [11]). *Let $f, g \in \mathcal{D}_y(\overline{U}, \mathbb{R}^n)$ and*

$$|f(x) - g(x)| < \operatorname{dist}(y, f(\partial U)) \qquad (x \in \partial U).$$

Then

$$\deg(f, U, y) = \deg(g, U, y).$$

For $C = (c_1, \cdots, c_n) \in \mathbb{R}^n$, we write $|C|_\infty = \max\big\{|c_i| : i = 1, \cdots, n\big\}$ and let $\Omega_c = \big\{C \in \mathbb{R}^n : |C|_\infty < c\big\}$. For an interval I and $f \in \mathcal{C}(I, \mathbb{R}^n)$, the norm $\|.\|$ is defined as

$$\|f\| = \max_{v \in I} |f(v)|_\infty. \tag{4}$$

If P is a matrix then $\|P\|$ denote it's usual norm.

2. Notations and Preliminaries

In this section, we give some sufficient conditions for the existence of periodic solution of the systems of the second and third order nonlinear autonomous differential equations in $\mathbb{R}^n$.

Theorem 2.1 (Emamirad-Mehri, [12]). *Consider the second order system*

$$X'' + F(X, X') = 0, \tag{5}$$

where X is a $\mathbb{R}^n$-valued function of t in $\mathbb{R}$ and n is a positive integer and for $F = (f_1, \cdots, f_n)^T$, in which the functions $f_1, \cdots, f_n$ are of the class $\mathcal{C}^2$ in a neighborhood of the origin in $\mathbb{R}^{2n}$. Consider $c > 0$, $k \in (1, 2)$ and a diagonal matrix A as:

$$A = \mathrm{diag}(a_1, \cdots, a_n), \qquad a_i > 0,$$
$$k \leq \frac{a_{\max}}{a_{\min}} < \frac{2}{k-1}, \quad a_{\min} = \min_{1 \leq i \leq n} \{a_i\}, \quad a_{\max} = \max_{1 \leq i \leq n} \{a_i\}.$$

Define,

$$D = \Big\{(X, X') : |X|_\infty \leq 2c, |X'|_\infty \leq 2ca_{\max}\Big\},$$
$$\omega_0 = \frac{\pi}{a_{\min} + a_{\max}}, \qquad \omega_1 = (k+1)\omega_0.$$

Now, if $\delta c > \frac{(k+1)M}{a_{\min}^2}$, where

$$\delta = \min\Big\{|\sin(a_i\omega_j)| : i = 1, \cdots, n, j = 0, 1\Big\},$$

$$M = \max\Big\{\big|A^2 X - F(X, X')\big|_\infty : (X, X') \in D\Big\},$$

then there exists ω, with $\omega_0 < \omega < \omega_1$ such that for some proper initial conditions, the system (5) has a solution which satisfies the following boundary conditions:

$$X(0) = X(\omega) = 0. \tag{6}$$

Corollary 2.2 (Bayat-Khatami, [4]). *Consider the fourth order differential equation (1). Assume that there exist $a, b, c > 0$, and $k \in (1, 2)$ such that*

$$k + \frac{1}{k} < \frac{a}{\sqrt{b}} < \frac{k-1}{2} + \frac{2}{k-1},$$
$$c\delta\beta^2 > M(k+1),$$

where

$$\delta = \min\left\{\sin\left(\frac{\alpha\pi}{\alpha+\beta}\right), \sin\left(\frac{\beta\pi}{\alpha+\beta}\right), \sin\left(\frac{\alpha\pi(k+1)}{\alpha+\beta}\right), \sin\left(\frac{\alpha\pi(k+1)}{\alpha+\beta}\right)\right\},$$
$$M = \max\Big\{\big|ax'' + bx - f(x, x', x'', x''')\big| : (x, x', x'', x''') \in D\Big\},$$
$$D = \Big\{(x, y, x', y') : |x| \leq 2c, |y| \leq 2c, |x'| \leq 2c\alpha, |y'| \leq 2c\alpha\Big\},$$
$$\alpha = \sqrt{\frac{a + \sqrt{a^2 - 4b}}{2}}, \qquad \beta = \sqrt{\frac{a - \sqrt{a^2 - 4b}}{2}}.$$

Then the equation (1) has a periodic solution.

The following lemma will be used in the next discussions.

Lemma 2.3. *Let*

$$A = \mathrm{diag}(a_1, a_2, \cdots, a_n), \qquad a_i > 0$$

be a positive diagonal matrix and $a_{\min} = \min\{a_i\}$ *and* $a_{\max} = \max\{a_i\}$*. If* $k \in (1,2)$ *and* $k \le \frac{a_{\max}}{a_{\min}} < \frac{2}{k-1}$*, then*

$$\deg\big(\cos(A\omega_j)C, \Omega_c, 0\big) = (-1)^j \qquad (j = 0, 1), \tag{7}$$

where

$$\omega_0 = \frac{\pi}{a_{\min} + a_{\max}}, \qquad \omega_1 = (k+1)\omega_0. \tag{8}$$

Proof. Using Definition 1.1, we have

$$\deg\big(\cos(A\omega_j)C, \Omega_c, 0\big) = \mathrm{sgn}\Big(\prod_{i=1}^{n} \cos(a_i\omega_j)\Big). \tag{9}$$

To prove (7), we have

$$\frac{(k-1)\pi}{k+1} < a_i\omega_0 < \frac{2\pi}{k+1} + \frac{\pi}{2},$$

and

$$\pi < a_i\omega_1 < 2\pi.$$

□

Now, we generalize Theorem 2.1 for third order systems as follows:

Theorem 2.4. *Consider the third order system*

$$X''' + F(X, X', X'') = 0, \tag{10}$$

where X *is a* $\mathbb{R}^n$*-valued function of* t *in* $\mathbb{R}$ *and* n *is a positive integer and for* $F = (f_1, \cdots, f_n)^T$*, the functions* $f_1, \cdots, f_n$ *are of class* $\mathcal{C}^2$ *in a neighborhood of the origin in* $\mathbb{R}^{3n}$*. Consider* $c > 0$ *and* $k \in (1,2)$ *with a diagonal matrix* A *as:*

$$\begin{aligned} &A = \mathrm{diag}(a_1, \cdots, a_n), \qquad a_i > 0, \\ &k \le \frac{a_{\max}}{a_{\min}} < \frac{2}{k-1}, \qquad a_{\min} = \min_{1\le i\le n}\{a_i\}, \qquad a_{\max} = \max_{1\le i\le n}\{a_i\}. \end{aligned} \tag{11}$$

Define,

$$D = \Big\{(X, X', X'') : |X|_\infty \le \frac{2c}{a_{\min}}, |X'|_\infty \le 2c, |X''|_\infty \le 2ca_{\max}\Big\},$$

$$\omega_0 = \frac{\pi}{a_{\min} + a_{\max}}, \quad \omega_1 = (k+1)\omega_0.$$

Now, if we have

$$\delta c a_{\min}^2 > \pi M(k+1), \tag{12}$$

where

$$\delta = \min\Big\{|\cos(a_i\omega_j)| : i = 1, \cdots, n, j = 0, 1\Big\},$$

$$M = \max\Big\{\big|A^2X' - F(X, X', X'')\big|_\infty : (X, X', X'') \in D\Big\},$$

then there exists ω with $\omega_0 < \omega < \omega_1$ such that for a proper initial conditions, the system (10) has a solution that satisfies the following boundary condition

$$X(0) = X(\omega). \tag{13}$$

Proof. Consider the following system:

$$\begin{cases} X''' + F(X, X', X'') = 0, \\ X(0) = -A^{-1}C, \\ X'(0) = 0, \\ X''(0) = AC, \end{cases} \tag{14}$$

where $|C|_\infty < c$ and X is a $\mathbb{R}^n$-valued function of t in $\mathbb{R}$. Now, we consider the following integral equation:

$$X\big(t, C, \lambda\big) = -A^{-1}\cos(At)C + \lambda G\big(t, X, X', X''\big), \tag{15}$$

$$G\big(t, X, X', X''\big) = A^{-2}\int_0^t \Big(\cos\big(A(t-s)\big) - I_n\Big)\Big[A^2 X'(s) - F\big(X(s), X'(s), X''(s)\big)\Big]ds.$$

One can easily verify that $X\big(t, C, 1\big)$ is the solution of (14) with $X\big(0, C, \lambda\big) = -A^{-1}C$ and this solution is not identically constant, since $X''\big(0\big) \neq 0$. The $X\big(t, C, 1\big)$ is the unique solution of (14) and the existence of such a solution for small $|t|$, follows from the standard existence theorem. Taking $|t|$ small enough, we can obtain the following estimates:

$$|X(t)|_\infty \le \frac{2c}{a_{\min}}, \qquad |X'(t)|_\infty \le 2c, \qquad |X''(t)|_\infty \le 2ca_{\max}.$$

Now, using the continuity theorem, we can extend this solution for the whole interval $I = \Big[0, \frac{(k+1)\pi}{2a_{\min}}\Big]$, and this solution will be continuous on the boundaries. Using (12), we have

$$\big\|G(t, X, X', X'')\big\| \le \frac{2tM}{a_{\min}^2} \le \frac{\pi M(k+1)}{a_{\min}^3} < \frac{\delta c}{a_{\min}} < \frac{c}{a_{\min}},$$

and thus

$$\big\|X(t, C, 1)\big\| \le \frac{2c}{a_{\min}}, \qquad \big\|X'(t, C, 1)\big\| \le 2c, \qquad \big\|X''(t, C, 1)\big\| \le 2ca_{\max}.$$

In the following, we show that this solution satisfies (13) for some ω where $\omega_0 < \omega < \omega_1$, or equivalently, we show that the number ω where $\omega_0 < \omega < \omega_1$ and the point C where $|C|_\infty = c$ exist such that $X(\omega, C, 1) = 0$. According to Lemma 2.3, we obtain

$$\deg\Big(X(\omega_j, C, 0), \Omega_c, 0\Big) = (-1)^j, \qquad (j = 0, 1). \tag{16}$$

On the other hand w have

$$\Big\|X(\omega_j, C, 1) - X(\omega_j, C, 0)\Big\| = \Big\|G(\omega_j, X, X', X'')\Big\| < \frac{\delta c}{a_{\min}}, \tag{17}$$

and also

$$\begin{aligned} \Big\|X(\omega_j, C, 0)\Big\| = \big|A^{-1}\cos(A\omega_j)C\big|_\infty &= \max_{i=1,\cdots,n}\Big\{\big|\cos(a_i\omega_j)\big|\,\frac{|c_i|}{a_i}\Big\} \\ &\ge \max_{i=1,\cdots,n}\Big(\frac{|c_i|}{a_i}\Big)\min_{i=1,\cdots,n}\big|\cos(a_i\omega_j)\big| > \frac{\delta c}{a_{\min}}. \end{aligned} \tag{18}$$

Thus by Theorem **??** and (16)-(18), it follows that

$$\deg\Big(X(\omega_j, C, 1), \Omega_c, 0\Big) = (-1)^j \qquad (j = 0, 1).$$

Now, by virtue of Theorem **??** it implies the existence of a number ω, $\omega_0 < \omega < \omega_1$ and a point C such that $|C|_\infty = c$ and $X(\omega, C, 1) = 0$. □

Corollary 2.5 (Bayat-Khatami, [5]). *Consider the fifth order differential equation (2). Assume that there exists $a, b, c > 0$ and $k \in (1, 2)$ such that*

$$k + \frac{1}{k} < \frac{a}{\sqrt{b}} < \frac{k-1}{2} + \frac{2}{k-1},$$

$$c\delta\beta^2 > (k+1)M\pi,$$

where

$$\delta = \min\left\{\cos\left(\frac{\alpha\pi}{\alpha+\beta}\right), \cos\left(\frac{\beta\pi}{\alpha+\beta}\right), \cos\left(\frac{\alpha\pi(k+1)}{\alpha+\beta}\right), \cos\left(\frac{\alpha\pi(k+1)}{\alpha+\beta}\right)\right\},$$

$$M = \max\Big\{\big|ax''' + bx' - f(x, x', x'', x''', x^{(4)})\big|: (x, x', x'', x''', x^{(4)}) \in D\Big\},$$

$$D = \Big\{(x, y, x', y', x'', y'') : |x| \leq \frac{2c}{\beta}, |y| \leq \frac{2c}{\beta}, |x'| \leq 2c, |y'| \leq 2c, |x''| \leq 2c\alpha, |y''| \leq 2c\alpha\Big\},$$

$$\alpha = \sqrt{\frac{a + \sqrt{a^2 - 4b}}{2}}, \qquad \beta = \sqrt{\frac{a - \sqrt{a^2 - 4b}}{2}}.$$

Then the equation (2) has a periodic solution.

Corollary 2.6. *Let F be an odd function of class $\mathcal{C}^2$ in a neighborhood of the origin in $\mathbb{R}^n$ and let $DF(0)$ be the derivative of F at the origin. If there exists a diagonal positive matrix A with $k \leq \frac{a_{\max}}{a_{\min}} < \frac{2}{k-1}$ for $k \in (1, 2)$, such that the norm of $DF(0) - A^2$ is sufficiently small, then for some ω in (ω_0, ω_1) these exists a 2ω-periodic solution for*

$$X''' + F(X') = 0.$$

Proof. According to Theorem 2.4, it is enough to show the smallness of the norm of matrix $DF(0) - A^2$. There are numbers $\varepsilon > 0$ and $d > 0$ such that for all $|X'|_\infty < 2d$, we have

$$\begin{aligned}\big|(DF(0) - A^2)X'\big|_\infty &\leq \big|DF(0) - A^2\big|_\infty |X'|_\infty - \varepsilon,\\ &\leq \frac{\delta a_{\max}^2}{(k+1)\pi}|X'|_\infty - \varepsilon,\end{aligned}$$

where $\delta = \cos\left(\frac{\pi(k+1)a_{\max}}{a_{\max}+a_{\min}}\right)$. From the differentiability of F at the origin it follows that for $\frac{\varepsilon}{2d} > 0$, there exists $\alpha > 0$ such that if $|X'|_\infty < 2\alpha$, then

$$\big|F(X') - A^2X'\big|_\infty \leq \frac{\varepsilon}{2d}|X'|_\infty + \big|DF(0)X' - A^2X'\big|_\infty.$$

Now for $c = \min\{\delta, d\}$ and $|X'|_\infty \leq 2c$,

$$M = \sup\Big\{\big|F(X') - A^2X'\big|_\infty : |X'|_\infty \leq 2c\Big\} \leq \frac{\delta a_{\max}^2 c}{(k+1)\pi}.$$

□

Remark 2.7. *In Theorem 2.4, if $A = aI_n$, then there is no such k that satisfies (11). For this purpose, we extend system (14) on $\mathbb{R}^{n+1}$ as follows:*

$$\widetilde{X}''' = -\widetilde{A}^2\widetilde{X}' + \widetilde{F}(\widetilde{X}, \widetilde{X}', \widetilde{X}''), \tag{19}$$

where

$$\widetilde{X}=\begin{bmatrix}X\\ y\end{bmatrix},\quad \widetilde{A}^2=\begin{bmatrix} & & & & -\varepsilon^2\\ & & & & 0\\ & a^2I_n & & & 0\\ & & & & \vdots\\ & & & & 0\\ k^2 & 0 & \cdots & 0 & 0\end{bmatrix}\quad \widetilde{F}=\begin{bmatrix}F_1-\varepsilon y^2\\ F_2\\ \vdots\\ F_n\\ 0\end{bmatrix},\quad F=\begin{bmatrix}F_1\\ F_2\\ \vdots\\ F_n\end{bmatrix}.$$

Using the transformation $\widetilde{X}=PU$, *the matrix* $\Lambda=P^{-1}\widetilde{A}^2P$ *is a diagonal matrix whose diagonal elements are eigenvalues of matrix* $\widetilde{A}^2$. *Thus, the new form of this system is:*

$$\widetilde{U}'''=-\Lambda U'+P^{-1}\widetilde{F}\big(PU,PU',PU''\big). \tag{20}$$

The diagonal elements of Λ *consist of the eigenvalues of* $\widetilde{A}^2$:

$$\lambda_1=\frac{a^2+d}{2},\qquad \lambda_{n+1}=\frac{a^2-d}{2},\qquad \lambda_i=a^2,\qquad (i=2,\cdots,n),$$

and $d=\sqrt{a^4-4\varepsilon^2k^2}$. *Now if* $2\varepsilon k<a^2<\frac{4}{\sqrt{3}}\varepsilon k$, *then these eigenvalues* λ_i *are real and* $\frac{\lambda\max}{\lambda_{\min}}$ *is less than* 4. *Also, the* P *matrix is given as follows:*

$$P=\begin{bmatrix}p & 0 & q\\ 0 & rI_{n-1} & 0\\ \frac{p\lambda_{n+1}}{\varepsilon^2} & 0 & -\frac{q\lambda_1}{\varepsilon^2}\end{bmatrix}\qquad (p,q,r\neq 0).$$

Moreover, the inverse matrix of P *is:*

$$P^{-1}=\begin{bmatrix}\frac{\lambda_1}{pa^2} & 0 & \frac{\lambda_{n+1}}{qa^2}\\ 0 & \frac{I_{n-1}}{r} & 0\\ \frac{\lambda_{n+1}}{qa^2} & 0 & \frac{-\varepsilon}{qa^2}\end{bmatrix}.$$

Now, we have to show:

$$\Big\|P^{-1}\widetilde{F}\big(PU,PU',PU''\big)\Big\|<\frac{\lambda_{n+1}}{3}\delta' c$$

where

$$\delta'=\min_{i=1,\cdots,n+1}\left|\cos\left(\frac{(k+1)\pi\sqrt{\lambda_i}}{a+\sqrt{\lambda_{n+1}}}\right)\right|$$

$$\widetilde{D}=\Big\{\big(\widetilde{X},\widetilde{X}',\widetilde{X}''\big)\colon |\widetilde{X}|_\infty\le\frac{2c}{a},|\widetilde{X}'|_\infty\le 2c,|\widetilde{X}''|_\infty\le 2ca\Big\}.$$

On the other hand, we have $\Big\|F\big(X,X',X''\big)\Big\|=M<\frac{a^2c}{3}$. *We can choose* ε^2 *small enough, such that*

$$\Big\|\widetilde{F}\big(\widetilde{X},\widetilde{X}',\widetilde{X}''\big)\Big\|=\widetilde{M}<\frac{a^2c}{3}.$$

Now, we have

$$\Big\|P^{-1}\widetilde{F}\big(PU,PU',PU''\big)\Big\|<\|P^{-1}\|\widetilde{M}<\|P^{-1}\|\frac{a^2c}{3},$$

which by choosing $\|P^{-1}\|<\frac{\delta'}{4}$, *for* p,q,r *being sufficiently large, the proof is complete.*

3. Main Results

In this section, we use Theorem 2.1 and Theorem 2.4 to show the existence of periodic solutions of certain higher order equations.

Theorem 3.1. *Consider the following equation*

$$x^{(2n)} + f\left(x, \cdots, x^{(2n-1)}\right) = 0, \tag{21}$$

where f is of class C^2 in a neighborhood of the origin in $\mathbb{R}^{2n}$. Suppose that there exist $a_i, i = 0, \cdots, n-1$ and a closed domain D containing the origin of $\mathbb{R}^{2n}$, such that

$$M := \left\{ \left| \sum_{i=0}^{n-1} a_i x^{(2i)} - f\left(x, x', \cdots, x^{(2n-1)}\right) \right| : \left(x, x', \cdots, x^{(2n-1)}\right) \in D \right\},$$

is sufficiently small. Also let

$$f\left(-x, x', -x'', \cdots, -x^{(2i)}, \cdots, x^{(2n-1)}\right) = -f\left(x, x', x'', \cdots, x^{(2i)}, \cdots, x^{(2n-1)}\right),$$

then the equation (21) has a periodic solution.

Proof. We consider the following two cases.
Case 1. Let n be an odd positive integer. The equation (21) can be rewritten to the following system

$$\begin{cases} x_1'' = -a^2 x_1 + \alpha x_2, \\ x_2'' = -a^2 x_2 + \alpha x_3, \\ \vdots \\ x_{n-1}'' = -a^2 x_{n-1} + \alpha x_n, \\ x_n'' = (n-1)a^2 x_n + \sum_{i=1}^{n-1} v_i x_i - \frac{1}{\alpha^{n-1}} f(.), \end{cases} \tag{22}$$

where

$$v_i = \binom{n}{i-1} \left(\frac{-1}{\alpha}\right)^{n-i} a^{2(n-i+1)},$$

with the arguments of $f(.)$ are as:

$$\begin{cases} x^{(2j)} = \sum_{i=1}^{j+1} w_{i-1,j} x_i \alpha^{i-1}, \\ x^{(2j+1)} = \left(x^{(2j)}\right)' \end{cases}$$

where

$$w_{ij} = \begin{cases} (-1)^{j-i} \binom{j}{i} a^{2(j-i)} & (i \leq j), \\ 0 & (i > j). \end{cases}$$

Now, if the function f is as:

$$f\left(x, x', \cdots, x^{(2n-1)}\right) = \sum_{i=0}^{n-1} \beta_i x^{(2i)} + g\left(x, x', \cdots, x^{(2n-1)}\right),$$

with β_i's satisfying the following conditions:

$$\begin{aligned} &\frac{2n-1}{2} a^2 < \beta_{n-1} < (n+1)a^2, \\ &\sum_{j=i}^{n-1} w_{i,j} \beta_j + (-1)^{n-i} \binom{n}{i} a^{2(n-i)} = 0, \qquad (i = 0, \cdots, n-2), \end{aligned}$$

then the system (22) can be rewritten as:

$$\begin{cases} x_1'' = -a^2 x_1 + \alpha x_2, \\ x_2'' = -a^2 x_2 + \alpha x_3, \\ \vdots \\ x_{n-1}'' = -a^2 x_{n-1} + \alpha x_n, \\ x_n'' = \Big((n-1)a^2 - \beta_{n-1}\Big) x_n - \frac{1}{\alpha^{n-1}} g(.). \end{cases} \tag{23}$$

Consider $c > 0, a > 0$ and $k \in (1, 2)$ and we define

$$\begin{aligned} X = (x_1, \cdots, x_n)^T, \qquad A = \operatorname{diag}\big(a, \cdots, a, \sqrt{\beta_{n-1} - (n-1)a^2}\big), \\ a_{\min} = \min\Big\{a, \sqrt{\beta_{n-1} - (n-1)a^2}\Big\} \\ a_{\max} = \max\Big\{a, \sqrt{\beta_{n-1} - (n-1)a^2}\Big\} \\ k \le \frac{a_{\max}}{a_{\min}} < \frac{2}{k-1} \\ \omega_0 = \frac{\pi}{a_{\min} + a_{\max}}, \qquad \omega_1 = (k+1)\omega_0 \\ D = \Big\{(X, X') : |X|_\infty \le 2c, |X'|_\infty \le 2c a_{\max}\Big\} \end{aligned} \tag{24}$$

with

$$\begin{cases} a_{\min}^2 > 2(k+1)\delta\alpha, \\ 2\delta\alpha^n c > (k+1)M, \end{cases}$$

where

$$\begin{aligned} M &= \max\Big\{\big|g(X, X')\big|_\infty : (X, X') \in D\Big\} \\ \delta &= \min\Big\{|\sin(a_i\omega_0)|, |\sin(a_i\omega_1)|\Big\}, \qquad a_i = a_{\min},\ a_{\max}. \end{aligned}$$

With α small enough, if M is sufficiently small, then by Theorem 2.1, there exists ω with $\omega_0 < \omega < \omega_1$ such that the system (23) has a solution that

$$X(0) = X(\omega) = 0.$$

So, for equation (21), we would have

$$x^{(2i)}(0) = x^{(2i)}(\omega) = 0, \qquad (i = 0, \cdots, n-1). \tag{25}$$

Now if in addition, we have

$$g\big(-x, x', -x'', x''', \cdots, -x^{(2n-2)}, x^{(2n-1)}\big) = -g\big(x, x', x'', x''', \cdots, x^{(2n-2)}, x^{(2n-1)}\big),$$

then the system of equation (23) has a periodic solution of period 2ω. For this, we can extend the obtained $x(t)$ with boundary condition (25) to $[0, 2\omega]$ as follows:

$$z(t) = \begin{cases} x(t) & (0 \le t \le \omega), \\ -x(2\omega - t) & (\omega \le t \le 2\omega). \end{cases}$$

This solution satisfies (21) and has continuous property at $t = \omega$ with

$$z^{(i)}(0) = z^{(i)}(\omega), \qquad (i = 0, \cdots, 2n-1).$$

Case 2. Let n be an even positive integer. The equation (21) can be rewritten to the following system

$$\begin{cases} x_1'' = -a^2 x_1 + \alpha x_2 + \alpha x_3, \\ x_2'' = -a^2 x_2 + \alpha x_1 + \alpha x_3, \\ \vdots \\ x_{n+1}'' = \big((n-1)a^2 - \alpha\big)x_{n+1} + \sum_{i=1}^{n} v_i^* x_i - \frac{1}{\alpha^{n-1}} f(.), \end{cases} \tag{26}$$

where

$$v_i^* = \begin{cases} v_i & (i \le 2), \\ v_{i-1} + v_i & (i > 2), \end{cases}$$

and the arguments of $f(.)$ are as:

$$\begin{cases} x^{(2j)} = \sum_{i=1}^{j+2} w_{i,j}^* x_i, \\ x^{(2j+1)} = \big(x^{(2j)}\big)', \end{cases}$$

where

$$w_{i,j}^* = \begin{cases} w_{i-1,j} & (i \le 2), \\ w_{i-2,j} + w_{i-1,j} & (i > 2). \end{cases}$$

Now, if the function f is as:

$$f\big(x, \cdots, x^{(2n-1)}\big) = \sum_{i=0}^{n-1} \beta_i x^{(2i)} + g\big(x, \cdots, x^{(2n-1)}\big),$$

with β_i's satisfying the following conditions:

$$\frac{2n-1}{2} a^2 < \beta_{n-1} + \alpha < (n+1)a^2,$$
$$v_i^* + S_i^* = 0,$$

where

$$S_i^* = \sum_{j=k-2}^{n-1} \beta_j w_{i,j},$$

then the system (26) can be rewritten as:

$$\begin{cases} x_1'' = -a^2 x_1 + \alpha x_2 + \alpha x_3, \\ x_2'' = -a^2 x_2 + \alpha x_1 + \alpha x_3, \\ \quad \vdots \\ x_{n+1}'' = \big((n-1)a^2 - \alpha - \beta_{n-1}\big)x_{n+1} - \frac{1}{\alpha^{n-1}} g(.). \end{cases} \tag{27}$$

Defining

$$\begin{cases} X = (x_1, \cdots, x_n, x_{n+1})^T \\ a_{\min} = \min\Big\{a, \sqrt{\beta_{n-1} - (n-1)a^2 + \alpha}\Big\} \\ a_{\max} = \max\Big\{a, \sqrt{\beta_{n-1} - (n-1)a^2 + \alpha}\Big\} \end{cases}$$

If we have

$$\begin{cases} a_{\min}^2 > 4(k+1)\delta\alpha, \\ 2\delta\alpha^n c > (k+1)M, \end{cases}$$

then by Theorem 2.1, there exists ω, $\omega_0 < \omega < \omega_1$ such that the system (26) has a solution that

$$X(0) = X(\omega) = 0.$$

And similar of Case 1, if the parity conditions hold then the equation (21) would has a periodic solution. □

Theorem 3.2. *Consider the following equation*

$$x^{(2n+1)} + f\big(x, \cdots, x^{(2n)}\big) = 0, \tag{28}$$

where f is of class $\mathcal{C}^2$ in a neighborhood of the origin in $\mathbb{R}^{2n+1}$. Suppose that there exist a, α and a closed domain containing the origin of $\mathbb{R}^{2n+1}$, such that

$$M := \max\Big\{\Big|\sum_{i=0}^{n-1} a_i x^{(2i+1)} - f\big(x, x', \cdots, x^{(2n)}\big)\Big| : \big(x, x', \cdots, x^{(2n)}\big) \in D\Big\},$$

is sufficiently small. Also let

$$f\big(x, -x', x'', \cdots, -x^{(2i+1)}, \cdots, x^{(2n)}\big) = -f\big(x, x', x'', \cdots, x^{(2i)}, \cdots, x^{(2n)}\big),$$

then equation (28) has a periodic solution.

Proof. We consider the following two cases.
Case 1. Let n be an odd positive integer. We can transfer this equation to the following third order system:

$$\begin{cases} x_1''' = -a^2 x_1' + \alpha x_2', \\ x_2''' = -a^2 x_2' + \alpha x_3', \\ \vdots \\ x_{n-1}''' = -a^2 x_{n-1}' + \alpha x_n', \\ x_n''' = (n-1)a^2 x_n' - \sum_{i=1}^{n-1} v_i x_i' - \frac{1}{\alpha^{n-1}} f(.), \end{cases} \tag{29}$$

where the arguments of the function $f(.)$ are as:

$$\begin{cases} x^{(2j+1)} = \sum_{i=1}^{j+1} w_{i-1,j} x_i', \qquad (j = 0, \cdots, n-1) \\ x^{(2j+2)} = \big(x^{(2j+1)}\big)'. \end{cases}$$

Now if the function f is as:

$$f\big(x, \cdots, x^{(2n)}\big) = \sum_{i=0}^{n-1} \beta_i x^{(2i+1)} + g\big(x, \cdots, x^{(2n)}\big),$$

with β_i's satisfying the following conditions:

$$\begin{aligned} &\frac{2n-1}{2} a^2 < \beta_{n-1} < (n+1)a^2, \\ &\sum_{j=i}^{n-1} \alpha^{-i} w_{i-1,j} \beta_j + (-1)^{n-i} \binom{n}{i} a^{2(n-i)} = 0, \qquad (i = 1, \cdots, n-2), \end{aligned}$$

then the system (29) can be rewritten as:

$$\begin{cases} x_1''' = -a^2 x_1' + \alpha x_2', \\ x_2''' = -a^2 x_2' + \alpha x_3', \\ \vdots \\ x_{n-1}''' = -a^2 x_{n-1}' + \alpha x_n', \\ x_{n+1}''' = \big((n-1)a^2 - \beta_{n-1}\big)x_n' - \frac{1}{\alpha^{n-1}} g(.), \end{cases} \tag{30}$$

If similar conditions to which presented in Case 1 hold with herein

$$D = \Big\{(X, X', X'') : |X|_\infty \le \frac{2c}{a_{\min}}, |X'|_\infty \le 2c, |X''|_\infty \le 2ca_{\max}\Big\}$$
$$M = \max\Big\{\big|g(X, X', X'')\big|_\infty : (X, X', X'') \in D\Big\},$$

then by Theorem 2.4, there exists ω with $\omega_0 < \omega < \omega_1$ such that the system (30) has a solution that

$$X(0) = X(\omega).$$

So, for the equation (28), we would have:

$$x^{(2i+1)}(0) = x^{(2i+1)}(\omega), \qquad (i = 0, \cdots, n-1). \tag{31}$$

Now if in addition, we have

$$g\big(x, -x', x'', -x''', \cdots, -x^{(2n-1)}, x^{(2n)}\big) = -g\big(x, x', x'', x''', \cdots, x^{(2n-1)}, x^{(2n)}\big),$$

then the equation (28) has a periodic solution of period 2ω. For this, we can extend the obtained solution $x(t)$ with the boundary conditions (31) to $[0, 2\omega]$ as follows:

$$z(t) = \begin{cases} x(t) & (0 \le t \le \omega), \\ \\ x(2\omega - t) & (\omega \le t \le 2\omega). \end{cases}$$

This solution satisfies (28) and has continuous property $t = \omega$ with

$$z^{(i)}(0) = z^{(i)}(\omega), \qquad (i = 0, \cdots, 2n).$$

Case 2. Let n be an even positive integer. We can transfer the equation (28) to the following system

$$\begin{cases} x_1''' = -a^2 x_1' + \alpha x_2' + \alpha x_3', \\ x_2''' = -a^2 x_2' + \alpha x_1' + \alpha x_3', \\ \vdots \\ x_{n+1}''' = \big((n-1)a^2 - \alpha\big)x_{n+1}' + \sum_{i=1}^{n} v_i^* x_i' - \frac{1}{\alpha^{n-1}} f(.), \end{cases} \tag{32}$$

where the arguments of $f(.)$ are as:

$$\begin{cases} x^{(2j+1)} = \sum_{i=1}^{j+2} w_{i,j}^* x_i', \\ x^{(2j+2)} = \big(x^{(2j+1)}\big)'. \end{cases}$$

Now if the function f is as:

$$f\big(x, \cdots, x^{(2n)}\big) = \sum_{i=0}^{n-1} \beta_i x^{(2i+1)} + g\big(x, \cdots, x^{(2n-1)}\big),$$

with β_i's satisfying the following conditions:

$$\frac{2n-1}{2}a^2 < \beta_{n-1} + \alpha < (n+1)a^2,$$
$$v_i^* + S_i^* = 0,$$

then the system (32) can be rewritten as:

$$\begin{cases} x_1''' = -a^2 x_1' + \alpha x_2' + \alpha x_3', \\ x_2''' = -a^2 x_2' + \alpha x_1' + \alpha x_3', \\ \vdots \\ x_{n+1}''' = \big((n-1)a^2 - \alpha\big)x_{n+1}' - \frac{1}{\alpha^{n-1}} f(.). \end{cases}$$

If similar conditions to those presented in Case 1 hold and D and M as in Case 2, then the existence of a ω with $\omega_0 < \omega < \omega_1$ such that $X(0) = X(\omega)$ is guaranteed. Furthermore if the equation (28) satisfies the suitable parity conditions similar to dose of Case 2 in Theorem 2.4, then the existence of a periodic solution for the equation is guaranteed. □

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