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Reconstruct the unknown source for the fractional elliptic equations : Regularization method and error estimates

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Abstract

The paper discusses the inverse problem of determining an unknown source term in a fractional elliptic equation in bounded domain. In order to solve the considered problem, a fractional Tikhonov is used. Applying this method, having a regularized solution is constructed. An a priori and a posteriori error estimates are obtained, and the the terminal data has a random data is considered.

Keywords: Regularization method, Fractional pseudo-parabolic problem, Ill-posed problem, Nonlocal problem, Convergence estimates.
2010 MSC: 35K05, 35K99, 47J06, 47H10x .

1. Introduction

Fractional models have recently become a subject of interest for many scientists because of their important applications in various fields, [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. In this work, we consider a source identification problem in a fractional elliptic partial differential equation as follows

$$\mathcal{D}_z^\beta \mathcal{D}_z^\beta u(z) - \mathcal{A}u(z) = \mathcal{F}, z > 0, \tag{1.1}$$

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where $0 < \beta < 1$ and $\mathcal{D}_z^\beta$ is the fractional Liouville-Caputo derivative of order β for differentiable function, defined by [13].

$$\mathcal{D}_z^\beta u(z,x) = \frac{1}{\Gamma(1-\beta)} \int_0^z (z-s)^{-\beta} u_s(s,x) ds, \quad 0 < \beta < 1. \tag{1.2}$$

We want to recover the unknown function f from the knowledge of the interior information

$$u(\mathcal{L}) = \rho \in \mathcal{H}, 0 < \mathcal{L} < \infty. \tag{1.3}$$

Here, $\mathcal{A} : D(\mathcal{A}) \in \mathcal{H} \to \mathcal{H}$ is a positive, self-adjoint operator with compact resolvent, and $\mathcal{H}$ denotes a separable Hilbert space endowed with the inner product $(.,.)$ and the norm $\|.\|$. Let (λ_n, e_n) be the eigenvalues and corresponding eigenfunctions of $\mathcal{A}$, such that $\{\lambda_n\}_{n\geq 1}$ is an increasing unbounded sequence and $\{e_n\}_{n\geq 1}$ forms an orthonormal basis in $\mathcal{H}$. The additional data ρ is observed at the position $z = \mathcal{L}$, which may contain measurement errors. The famous difficulty with inverse problems is that they are often ill-posed. Thus, Therefore, it is necessary to correct this problem and provide error estimates.

In [14], the elliptic fractional operator $(\mathcal{D}_z^\beta \mathcal{D}_z^\beta - \mathcal{A})$ is subjected to an inverse problem in an infinite domain, the authors proposed a preconditioning version of the Kozlov-Maz'ya iteration method for recovering missing data under a complementary condition. In the same context, several publications have been published on ill-posed inverse fractional problems, using a variety of regularization methods to overcome the ill-posedness, see in [17, 18]. In [19, 20], they obtained the regularized sought solution by using quasi-boundary value method, where in [[21]] they implemented the quasireversibility method, in [22] the authors proposed the Fourier truncation method, also, in [23], a non-stationary iterative Tikhonov regularisation method coupled with a finite dimensional approximation is applied to recover a stable source term.

In this work, we provide a fractional Tikhonov method to solve this inverse source problem for the fractional elliptic diffusion equation in a general bounded domain. This method was first proposed by Li and Xiong [16] when they considered an inverse heat conduction problem. Afterthat, this method to solve a Cauchy problem of the Helmholtz equation.

In this paper, since the a-priori bound of the exact solution is usually hard for one to estimate, the a-priori parameter choice rule is unavailable. In this paper, we will give the convergence rates under the a-priori parameter choice rule and the a-posteriori parameter choice rule. We want to recover the source function $\mathcal{F}(x)$ from indirect observable data $u(\mathcal{L}) = \rho$ at the final moment $z = \mathcal{L}$. The observable data $\rho(x)$ contain measurement errors and satisfies

$$\left\|\rho^\epsilon - \rho\right\|_{\mathcal{H}} \leq \epsilon. \tag{1.4}$$

unless otherwise specified, in this paper, $\|\cdot\|$ is the L^2 norm and $\epsilon > 0$ is the noise level. Next, see in [15], we have

$$\rho(x_k) = \rho(x_k) + \varepsilon_k, \quad k = 1, \cdots, n.$$

where $\epsilon_k, k = 1, \cdots, n$ are unknown independent random errors because the function $\rho(x)$ in practical applications is the result of experimental observations and cannot be viewed without errors. As a matter of fact, these mistakes can emerge out of many sources like the estimating instrument or the climate. From now on, we put $x_k = \pi\frac{2k-1}{2n}$, with $k = 1, \cdots, n$. We have a data set $\mathcal{D} = \big(\widetilde{\rho}(x_1), \widetilde{\rho}(x_2), \ldots, \widetilde{\rho}(x_n)\big)$, which is the measure of $\big(\rho(x_1), \rho(x_2), \ldots, \rho(x_n)\big)$, here $\mathcal{D}$ satisfies

$$\widetilde{\rho}\big(x_k\big) = \rho\big(x_k\big) + \sigma_k \epsilon_k, \tag{1.5}$$

where, $\epsilon_k,\ k = 1, \cdots, n$ are unknown independent noises. Therefore, ϵ_k and σ_k are unknown positive constants that are constrained by the positive constant $\mathcal{V}_{max}$ so that $0 \leq \sigma_k$ and $\mathcal{V}_{\max}$, respectively. The noises ε_k are independent of one another.

The outline of the paper is as follows. In Section 2, we recall the necessary tools to treat the considered problem. The mild solution is shown in Section 3. In Section 4, we get the non-well posed of problem (1.1), establishing the convergence estimates with a priori and a posteriori regularization parameter choice rule, by using a fractional Tikhonov method in Section 5. Finally, in section 6, showing the convergent rate under case the function ρ has a random data.

2. Preliminary

It is well known that the classical Mittag-Leffler function is one of the basic tools in fractional calculus, denotes $E_{\beta,1}(\cdot)$ and is defined by

$$E_{\beta,1}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(1+n\beta)}, \beta > 0, z \in \mathbb{C}. \tag{2.6}$$

Theorem 2.1. *For every $\beta \in (0,1)$, we have*

$$\frac{1}{1+\Gamma(1-\beta)x} \leq E_{\beta,1}(-x) \leq \frac{1}{1+\Gamma(1+\beta)^{-1}x}, x \geq 0. \tag{2.7}$$

From (2.7), we deduce that

$$\frac{1}{1+\Gamma(1-\beta)\sqrt{\lambda_n}\mathcal{L}^\beta} \leq E_{\beta,1}\big(-\sqrt{\lambda_n}\mathcal{L}^\beta\big) \leq \frac{1}{1+\Gamma(1+\beta)^{-1}\sqrt{\lambda_n}\mathcal{L}^\beta}, \mathcal{L} > 0. \tag{2.8}$$

3. The mild solution

For $0 < \beta < 1$, let us consider the following well-posed system equations

$$\begin{cases} \mathcal{D}_z^\beta \mathcal{D}_z^\beta u(z) - \mathcal{A}u(z) = \mathcal{F}, & z \in (0,\infty), \\ u(0) = 0. \end{cases} \tag{3.9}$$

Theorem 3.1. *Let $\mathcal{F} \in \mathcal{H}$, then the problem (1.1) admits a unique generalized solution*

$$\begin{aligned} u(z) &= -\big(I - E_{\beta,1}\big(-z^\beta\sqrt{\mathcal{A}}\big)\big)\mathcal{A}^{-1}\mathcal{F} = -\mathcal{K}_{\beta,1}(z)\mathcal{F} \\ &= -\sum_{k=1}^{+\infty} \frac{\big(1 - E_{\beta,1}\big(-\sqrt{\lambda_k}\ z^\beta\big)\big)}{\lambda_k} \langle \mathcal{F}, e_k \rangle e_k. \end{aligned} \tag{3.10}$$

Let $z = \mathcal{L}$ in (3.10), we obtain

$$u(\mathcal{L}) = -\big(I - E_{\beta,1}\big(-\mathcal{L}^\beta\sqrt{\mathcal{A}}\big)\big)\mathcal{A}^{-1}\mathcal{F} = -\mathcal{K}_{\beta,1}(\mathcal{L})\mathcal{F} = \rho. \tag{3.11}$$

$\mathcal{K}_{\beta,1}(\mathcal{L})$ is a self-adjoint compact linear operator and $\sup_{z\geq 0} \big\|\mathcal{K}_{\beta,1}(z)\big\| \leq \lambda_1^{-1}$. *For $\rho \in \mathcal{H}$, the space $\mathcal{H}^1$ is defined by*

$$\mathcal{H}^1 = \Big\{\rho \in \mathcal{H} : \|\mathcal{A}\rho\|_{\mathcal{H}} < \infty\Big\}. \tag{3.12}$$

The operator equation (3.11) admits a unique solution if and only if $g \in \mathcal{H}^1$.

4. Ill-posedness of the inverse problem (1.1)

To determine the unknown function $\mathcal{F}$, we just need to solve The operator equation (3.11), then we have the following

$$\begin{aligned}\mathcal{F} = -\mathcal{K}_\beta^{-1}(\mathcal{L})\rho &= -\mathcal{A}\big(I - E_{\beta,1}\big(-\mathcal{L}^\beta\sqrt{\mathcal{A}}\big)\big)^{-1}\rho \\ &= \sum_{k=1}^{+\infty} \frac{-\lambda_k}{1-\big(E_{\beta,1}\big(-\mathcal{L}^\beta\sqrt{\lambda_k}\big)\big)}\langle \rho, e_k\rangle e_k, \end{aligned} \tag{4.13}$$

We can see from (4.13) that the terms $\frac{\lambda_k}{(1-E_{\beta,1}(-\mathcal{L}^\beta\sqrt{\lambda_k}))}$ are the instability causes. From (2.8), we get

$$\lambda_k^{-\frac{1}{2}}\mathcal{C}_1(\beta) \le E_{\beta,1}\big(-\mathcal{L}^\beta\sqrt{\lambda_k}\big) \le \mathcal{C}_2(\beta), \tag{4.14}$$

which implies that

$$1-\mathcal{C}_2(\beta) \le 1-E_{\beta,1}\big(-\mathcal{L}^\beta\sqrt{\lambda_k}\big) \le 1-\mathcal{C}_1(\beta)\lambda_k^{-\frac{1}{2}}. \tag{4.15}$$

and so

$$\lambda_k \le \frac{\lambda_k}{1-\mathcal{A}_1(\beta)\lambda_k^{-\frac{1}{2}}} \le \frac{\lambda_k}{1-E_{\beta,1}\big(-\mathcal{L}^\beta\sqrt{\lambda_k}\big)} \le \frac{\lambda_k}{1-\mathcal{C}_2(\beta)}, \tag{4.16}$$

and therefore

$$\lambda_k \le \frac{\lambda_k}{1-E_{\beta,1}\big(-\mathcal{L}^\beta\sqrt{\lambda_k}\big)} \le \frac{\lambda_k}{1-\mathcal{C}_2(\beta)} \to \infty \text{ as } k\to\infty, \tag{4.17}$$

Theorem 4.1. *Let the following condition holds*

$$\|\mathcal{F}\|^2_{\mathcal{H}^\theta} = \sum_{k=1}^{+\infty}\lambda_k^{2\theta}\big|\langle\mathcal{F}, e_k\rangle\big|^2 \le E^2, \theta>0, \tag{4.18}$$

then

$$\|\mathcal{F}\|_{L^2(\Omega)} \le \mathcal{C}_\theta E^{\frac{1}{1+\theta}}\|\rho\|^{\frac{\theta}{1+\theta}}_{L^2(\Omega)}, \tag{4.19}$$

where $\mathcal{C}_\theta = \big(1-\mathcal{C}_2(\beta)\big)^{-\frac{\theta}{1+\theta}}$.

Proof. See proof in [13]. □

5. Fractional Tikhonov Regularization method

In this section, we propose a fractional Tikhonov regularization method to solve the ill-posed problem (1.1) and prove convergence estimate under the a-priori regularization parameter choice rule. The solutions of fractional Tikhonov regularization method with noisy data and exact data are given by

$$\mathcal{F}^\epsilon_{[\gamma(\epsilon)]}(x) = \sum_{k=1}^{+\infty} -\frac{\big|\mathcal{K}_{\beta,1}(\mathcal{L})\big|^{2\alpha-1}}{\big|\mathcal{K}_{\beta,1}(\mathcal{L})\big|^{2\alpha}+\gamma(\epsilon)}\langle\rho^\epsilon, e_k\rangle e_k(x), \quad \frac{1}{2}\le\alpha\le 1, \tag{5.20}$$

and

$$\mathcal{F}_{[\gamma(\epsilon)]}(x) = \sum_{k=1}^{+\infty} -\frac{\big|\mathcal{K}_{\beta,1}(\mathcal{L})\big|^{2\alpha-1}}{\big|\mathcal{K}_{\beta,1}(\mathcal{L})\big|^{2\alpha}+\gamma(\epsilon)}\langle\rho, e_k\rangle e_k(x), \quad \frac{1}{2}\le\alpha\le 1, \tag{5.21}$$

respectively, where $\gamma(\epsilon) > 0$ plays the role of regularization parameter and α is called the fractional parameter.

- $\alpha = \frac{1}{2}$, it is the quasi-boundary value method.
- $\alpha = 1$, it is the classic Tikhonov method.
- However, for $\frac{1}{2} < \alpha < 1$, the fractional Tikhonov method looks like between the quasi-boundary value method and the classic Tikhonov method.

Lemma 5.1. *[16] For constants $r \geq \lambda_1$ and $\frac{1}{2} \leq \alpha \leq 1$, we have*

$$\mathcal{G}_1(r) = \frac{r}{\mathcal{C}_\beta^{2\alpha} + \gamma(\epsilon) r^{2\alpha}} \leq \mathcal{A}_1(\alpha, \mathcal{C}_\beta^{2\alpha})[\gamma(\epsilon)]^{-\frac{1}{2\alpha}}, \tag{5.22}$$

where $\mathcal{A}_1 = \mathcal{A}_1(\alpha, \mathcal{C}_\beta) > 0$ are independent on α, s.

Lemma 5.2. *[16] For the constants $r \geq \lambda_1 > 0$ and $\frac{1}{2} \leq \alpha \leq 1$, we have*

$$\mathcal{G}_2(r) = \frac{\gamma(\epsilon) r^{2\alpha-\theta}}{\mathcal{C}_\beta^{2\alpha} + [\gamma(\epsilon)] r^{2\alpha}} \leq \begin{cases} \mathcal{A}_2 \mu^{\frac{\theta}{2\alpha}}, & 0 < \theta < 2\alpha, \\ \mathcal{A}_3[\gamma(\epsilon)], & \theta \geq 2\alpha, \end{cases} \tag{5.23}$$

where $\mathcal{A}_2 = \mathcal{A}_2(\alpha, \theta, \mathcal{C}_\beta) > 0, \mathcal{A}_3 = \mathcal{A}_3(\alpha, \theta, \lambda_1) > 0$ are independent on r.

Lemma 5.3. *[16] For constants $r \geq \lambda_1 > 0$ and $\frac{1}{2} < \alpha \leq 1$, we have*

$$\mathcal{G}_3(r) = \frac{\gamma(\epsilon) r^{2\alpha-1-\theta}}{\mathcal{C}_\beta^{2\alpha} + [\gamma(\epsilon)] r^{2\alpha}} \leq \begin{cases} \mathcal{A}_4[\gamma(\epsilon)]^{\frac{\theta+1}{2\alpha}}, & 0 < \theta < 2\alpha - 1, \\ \mathcal{A}_5[\gamma(\epsilon)], & \theta \geq 2\alpha - 1, \end{cases} \tag{5.24}$$

where $c_4 = c_4(\alpha, \theta, \mathcal{C}_\beta) > 0, \mathcal{A}_5 = \mathcal{A}_5(\alpha, \theta, \lambda_1, \mathcal{C}_\beta) > 0$.

Theorem 5.1. *Suppose the a-priori condition* (4.18) *and the noise assumption* (1.4) *hold, then*

- *If $0 < \theta < 2\alpha$ and $\gamma(\epsilon) = \left(\frac{\epsilon}{E}\right)^{\frac{2\alpha}{\theta+1}}$, we have*

$$\left\| \mathcal{F}^\epsilon_{[\gamma(\epsilon)]} - \mathcal{F} \right\|_2 \ \textit{is of order} \ \epsilon^{\frac{\theta}{\theta+1}}. \tag{5.25}$$

- *If $\theta \geq 2\alpha$ and $\gamma(\epsilon) = \left(\frac{\epsilon}{E}\right)^{\frac{2\alpha}{2\alpha+1}}$, we have*

$$\left\| \mathcal{F}^\epsilon_{[\gamma(\epsilon)]} - \mathcal{F} \right\| 2 \ \textit{is of order} \ \epsilon^{\frac{2\alpha}{2\alpha+1}}. \tag{5.26}$$

Proof. We know that

$$\left\| \mathcal{F}^\epsilon_{[\gamma(\epsilon)]} - \mathcal{F} \right\|_2 \leq \left\| \mathcal{F}^\epsilon_{[\gamma(\epsilon)]} - \mathcal{F}_{[\gamma(\epsilon)]} \right\|_2 + \left\| \mathcal{F} - \mathcal{F}_{[\gamma(\epsilon)]} \right\|_2 = \mathcal{I}_1 + \mathcal{I}_2. \tag{5.27}$$

For $k \geq 1$ and $\alpha > 0$, we have $\lambda_k \geq \lambda_1 > 1$. Thus

$$1 - E_{\beta,1}(-\mathcal{L}^\beta \sqrt{\lambda_k}) \geq 1 - E_{\beta,1}\big(-\mathcal{L}^\beta \sqrt{\lambda_1}\big) = \mathcal{C}_\beta. \tag{5.28}$$

Estimate to $\mathcal{I}_1$, by Lemma 5.1 and (5.28) we have

$$\begin{aligned} \mathcal{I}_1 = \left\| \mathcal{F}^\epsilon_{[\gamma(\epsilon)]} - \mathcal{F}_{[\gamma(\epsilon)]} \right\|_2 &= \left\| \sum_{k=1}^{+\infty} \frac{\left|\mathcal{K}_{\beta,1}(\mathcal{L})\right|^{2\alpha-1}}{\left|\mathcal{K}_{\beta,1}(\mathcal{L})\right|^{2\alpha} + \gamma(\epsilon)} \langle \rho^\epsilon - \rho, e_k \rangle e_k(x) \right\|_2 \\ &\leq \epsilon \sup_{k \in \mathbb{N}} \frac{\left|\mathcal{K}_{\beta,1}(\mathcal{L})\right|^{2\alpha-1}}{\left|\mathcal{K}_{\beta,1}(\mathcal{L})\right|^{2\alpha} + [\gamma(\epsilon)]} \\ &\leq \epsilon \sup_{k \in \mathbb{N}} \frac{\lambda_k}{\mathcal{C}_\beta^{2\alpha} + [\gamma(\epsilon)] \lambda_k^{2\alpha}} \leq \mathcal{A}_1 \epsilon [\gamma(\epsilon)]^{-\frac{1}{2\alpha}}. \end{aligned} \tag{5.29}$$

Next, estimate of $\mathcal{I}_2$, by (5.2), it yields

$$\begin{aligned}\mathcal{I}_2 &= \left\|\mathcal{F} - \mathcal{F}_{[\gamma(\epsilon)]}\right\|_2 \\ &= \Big\| \sum_{k=1}^{+\infty} \Big(\frac{\left|\mathcal{K}_{\beta,1}(\mathcal{L})\right|^{2\alpha-1}}{\left|\mathcal{K}_{\beta,1}(\mathcal{L})\right|^{2\alpha} + [\gamma(\epsilon)]} \langle \rho, e_k \rangle - \frac{1}{\left|\mathcal{K}_{\beta,1}(\mathcal{L})\right|} \langle \rho, e_k \rangle \Big) e_k(x) \Big\|_2 \\ &= \Big\| \sum_{k=1}^{+\infty} \frac{\gamma(\epsilon)\lambda_k^{-\theta}}{\left|\mathcal{K}_{\beta,1}(\mathcal{L})\right|^{2\alpha} + \gamma(\epsilon)} \cdot \frac{\lambda_k^{\theta}\langle \rho, e_k \rangle}{\left|\mathcal{K}_{\beta,1}(\mathcal{L})\right|} e_k(x) \Big\|_2 \le E \sup_{k\in\mathbb{N}} \frac{[\gamma(\epsilon)]\lambda_k^{-\theta}}{\left|\mathcal{K}_{\beta,1}(\mathcal{L})\right|^{2\alpha} + \gamma(\epsilon)} \\ &\le E \sup_{k\in\mathbb{N}} \frac{\gamma(\epsilon)\lambda_k^{2\alpha-\theta}}{\mathcal{C}_\beta^{2\alpha} + [\gamma(\epsilon)]\lambda_k^{2\alpha}} \le \begin{cases} \mathcal{A}_2 E[\gamma(\epsilon)]^{\frac{\theta}{2\alpha}}, & 0 < \theta < 2\alpha, \\ \mathcal{A}_3 E\gamma(\epsilon), & \theta \ge 2\alpha. \end{cases}\end{aligned}$$

From estimation of $\mathcal{I}_1$ and $\mathcal{I}_2$, we obtain

$$\left\|\mathcal{F}^{\epsilon}_{[\gamma(\epsilon)]} - \mathcal{F}\right\|_2 \le \mathcal{A}_1 \epsilon [\gamma(\epsilon)]^{-\frac{1}{2\alpha}} + \begin{cases} \mathcal{A}_2 E[\gamma(\epsilon)]^{\frac{\theta}{2\alpha}}, & 0 < \theta < 2\alpha, \\ \mathcal{A}_3 E[\gamma(\epsilon)], & \theta \ge 2\alpha. \end{cases} \tag{5.30}$$

Choose the regularization parameter $[\gamma(\epsilon)]$ by

$$\gamma(\epsilon) = \begin{cases} \left(\frac{\epsilon}{E}\right)^{\frac{2\alpha}{\theta+1}}, & 0 < \theta < 2\alpha, \\ \left(\frac{\epsilon}{E}\right)^{\frac{2\alpha}{2\alpha+1}}, & \theta \ge 2\alpha. \end{cases} \tag{5.31}$$

Then

$$\left\|\mathcal{F}^{\epsilon}_{[\gamma(\epsilon)]} - \mathcal{F}\right\|_2 \le \begin{cases} (\mathcal{A}_1 + \mathcal{A}_2)\, \epsilon^{\frac{\theta}{\theta+1}} E^{\frac{1}{\theta+1}}, & 0 < \theta < 2\alpha, \\ (\mathcal{A}_1 + \mathcal{A}_3)\, \epsilon^{\frac{2\alpha}{2\alpha+1}} E^{\frac{1}{2\alpha+1}}, & \theta \ge 2\alpha. \end{cases} \tag{5.32}$$

The proof is completed. □

5.1. The a-posteriori parameter choice rule

The most general a-posteriori rule is the Morozov's discrepancy principle. Here, the Morozov's discrepancy principle is used to determine the regularization parameter $\gamma(\epsilon)$. Using the discrepancy principle in the following form:

$$\left\| \frac{\left|\mathcal{K}_{\beta,1}(\mathcal{L})\right|^{2\alpha}}{\left|\mathcal{K}_{\beta,1}(\mathcal{L})\right|^{2\alpha} + [\gamma(\epsilon)]} \langle \rho^\epsilon - \rho^\epsilon, e_k \rangle \right\|_2 = \tau\epsilon. \tag{5.33}$$

where $\frac{1}{2} \le \alpha \le 1, \tau > 1$ are user-supplied constants which are independent on $\epsilon, \gamma > 0$ is the regularization parameter.

Lemma 5.4. *Let* $\theta(\mu) = \left\| \frac{|\mathcal{K}_{\beta,1}(\mathcal{L})|^{2\alpha}}{|\mathcal{K}_{\beta,1}(\mathcal{L})|^{2\alpha} + \gamma(\epsilon)} \langle \rho^\epsilon - \rho^\epsilon, e_k \rangle \right\|_2$, *then the following conclusions hold:*

- $\Theta(\gamma)$ *is a continuous function;*
- $\lim_{\gamma\to 0} \theta(\gamma) = 0$*;*
- $\lim_{\gamma\to\infty} \Theta(\gamma) = \left\|\rho^\epsilon\right\|$*;*
- $\Theta(\gamma)$ *is a strictly increasing function over* $(0, \infty)$.

Proof. The conclusions are straightforward if we note that

$$\theta([\gamma(\epsilon)]) = \left(\sum_{k=1}^{+\infty} \left(\frac{[\gamma(\epsilon)]}{|\mathcal{K}_{\beta,1}(\mathcal{L})|^{2\alpha} + [\gamma(\epsilon)]} \right)^2 \left| \langle \rho^\epsilon, e_k \rangle \right|^2 \right)^{\frac{1}{2}}. \tag{5.34}$$

□

Lemma 5.5. *If* $[\gamma(\epsilon)]$ *is the solution of Eq.* (5.34), *we see that*

$$[\gamma(\epsilon)]^{-\frac{1}{2\alpha}} \le \begin{cases} \left(\dfrac{\mathcal{A}_4^2}{\tau - 1} \right)^{\frac{1}{\theta+1}} \left(\dfrac{E}{\epsilon} \right)^{\frac{1}{\theta+1}}, & 0 < \theta < 2\alpha - 1, \\ \left(\dfrac{\mathcal{A}_5^2}{\tau - 1} \right)^{\frac{1}{2\alpha}} \left(\dfrac{E}{\epsilon} \right)^{\frac{1}{2\alpha}}, & \theta \ge 2\alpha - 1. \end{cases} \tag{5.35}$$

Proof. From (5.34), we obtain

$$\tau\epsilon = \left\| \sum_{k=1}^{+\infty} \frac{[\gamma(\epsilon)]}{\left|\mathcal{K}_{\beta,1}(\mathcal{L})\right|^{2\gamma} + [\gamma(\epsilon)]} \langle \rho^\epsilon, e_k \rangle e_k(x) \right\|_2 \tag{5.36}$$

$$\le \left\| \sum_{k=1}^{+\infty} \frac{[\gamma(\epsilon)]}{\left|\mathcal{K}_{\beta,1}(\mathcal{L})\right|^{2\alpha} + [\gamma(\epsilon)]} \langle \rho^\epsilon - \rho, e_k \rangle e_k(x) \right\|_2 + \left\| \sum_{k=1}^{+\infty} \frac{[\gamma(\epsilon)]}{\left|\mathcal{K}_{\beta,1}(\mathcal{L})\right|^{2\alpha} + [\gamma(\epsilon)]} \langle \rho, e_k \rangle e_k(x) \right\|_2$$

$$\le \epsilon + \left\| \sum_{k=1}^{+\infty} \frac{[\gamma(\epsilon)]}{\left|\mathcal{K}_{\beta,1}(\mathcal{L})\right|^{2\alpha} + [\gamma(\epsilon)]} \left|\mathcal{K}_{\beta,1}(\mathcal{L})\right| \lambda_k^{-\theta} \lambda_k^{\theta} \langle \mathcal{F}, e_k \rangle e_k(x) \right\|_2 \tag{5.37}$$

$$\le \epsilon + E \sup_{k\in\mathbb{N}} \frac{[\gamma(\epsilon)]}{\left(\frac{\mathcal{C}_\beta}{\lambda_k}\right)^{2\alpha} + [\gamma(\epsilon)]} \lambda_n^{-1-\theta} \le \epsilon + E \sup_{k\in\mathbb{N}} \frac{[\gamma(\epsilon)]\lambda_k^{2\alpha-\theta-1}}{\mathcal{C}_\beta^{2\alpha} + [\gamma(\epsilon)]\lambda_k^{2\alpha}}. \tag{5.38}$$

According to Lemma 5.3, we have

$$\tau\epsilon \le \epsilon + E \begin{cases} \mathcal{A}_4[\gamma(\epsilon)]^{\frac{\theta+1}{2\alpha}}, & 0 < \theta < 2\alpha - 1, \\ \mathcal{A}_5[\gamma(\epsilon)], & \theta \ge 2\alpha - 1. \end{cases} \tag{5.39}$$

So

$$[\gamma(\epsilon)]^{-\frac{1}{2\alpha}} \le \begin{cases} \left(\frac{\mathcal{A}_4}{\tau-1} \right)^{\frac{1}{\theta+1}} \left(\frac{E}{\epsilon} \right)^{\frac{1}{\theta+1}}, & 0 < \theta < 2\alpha - 1, \\ \left(\frac{\mathcal{A}_5}{\tau-1} \right)^{\frac{1}{2\alpha}} \left(\frac{E}{\epsilon} \right)^{\frac{1}{2\alpha}}, & \theta \ge 2\alpha - 1. \end{cases} \tag{5.40}$$

□

Theorem 5.2. *Suppose the a-priori condition* (4.18) *and the noise assumption* (1.4) *hold, then,*

- *If* $0 < \theta < 2\alpha - 1$, *we have*

$$\left\| \mathcal{F}^\epsilon_{[\gamma(\epsilon)]} - \mathcal{F} \right\|_2 \le \left(\mathcal{A}_1 \left(\frac{\mathcal{A}_4}{\tau - 1} \right)^{\frac{1}{\theta+1}} + \left(\frac{\tau + 1}{\mathcal{C}_\beta} \right)^{\frac{\theta}{\theta+1}} \right) E^{\frac{1}{\theta+1}} \epsilon^{\frac{\theta}{\theta+1}}. \tag{5.41}$$

- *If* $\theta \ge 2\alpha - 1$, *we have*

$$\left\| \mathcal{F}^\epsilon_{[\gamma(\epsilon)]} - \mathcal{F} \right\|_2 \le \left(\left(\frac{\mathcal{A}_5}{\tau - 1} \right)^{\frac{1}{2\alpha}} + \left(\frac{\tau + 1}{\mathcal{C}_\beta} \right)^{1-\frac{1}{2\alpha}} \lambda_1^{2\alpha-1-\theta} \right) E^{\frac{1}{2\alpha}} \epsilon^{1-\frac{1}{2\alpha}}. \tag{5.42}$$

Proof. By the triangle inequality, we know

$$\left\|\mathcal{F}^{\epsilon}_{[\gamma(\epsilon)]}-\mathcal{F}\right\|_2 \le \left\|\mathcal{F}^{\epsilon}_{[\gamma(\epsilon)]}-\mathcal{F}_{[\gamma(\epsilon)]}\right\|_2+\left\|\mathcal{F}-\mathcal{F}_{[\gamma(\epsilon)]}\right\|_2=\mathcal{I}_3+\mathcal{I}_4. \tag{5.43}$$

- For $0<\theta<2\alpha-1$, estimate of $\mathcal{I}_3$, by Lemma 5.3 we have

$$\mathcal{I}_3=\left\|\mathcal{F}^{\epsilon}_{[\gamma(\epsilon)]}-\mathcal{F}\right\|_2 \le \mathcal{A}_1\epsilon[\gamma(\epsilon)]^{-\frac{1}{2\alpha}} \le \mathcal{A}_1\left(\frac{\mathcal{A}_4}{\tau-1}\right)^{\frac{1}{\theta+1}} E^{\frac{1}{\theta+1}}\epsilon^{\frac{\theta}{\theta+1}}. \tag{5.44}$$

Now we estimate $\mathcal{I}_4$, we can deduce that

$$\begin{aligned}
\mathcal{I}_4=&\left\|\mathcal{F}-\mathcal{F}_{[\gamma(\epsilon)]}\right\|_2=\left\|\sum_{k=1}^{+\infty}\frac{[\gamma(\epsilon)]}{\left|\mathcal{K}_{\beta,1}(\mathcal{L})\right|^{2\gamma}+[\gamma(\epsilon)]}\big\langle\mathcal{F},e_k\big\rangle e_k(x)\right\|_2\\
=&\left\|\sum_{k=1}^{+\infty}\frac{[\gamma(\epsilon)]\left|\mathcal{K}_{\beta,1}(\mathcal{L})\right|}{\left|\mathcal{K}_{\beta,1}(\mathcal{L})\right|^{2\alpha}+[\gamma(\epsilon)]}\frac{\big\langle\mathcal{F},e_k\big\rangle}{\left|\mathcal{K}_{\beta,1}(\mathcal{L})\right|}e_k(x)\right\|_2\\
\le&\left\|\sum_{k=1}^{+\infty}\frac{[\gamma(\epsilon)]\left|\mathcal{K}_{\beta,1}(\mathcal{L})\right|}{\left|\mathcal{K}_{\beta,1}(\mathcal{L})\right|^{2\alpha}+[\gamma(\epsilon)]}\big\langle\mathcal{F},e_k\big\rangle e_k(x)\right\|_2^{\frac{\theta}{\theta+1}}\\
&\times\left\|\sum_{k=1}^{+\infty}\frac{[\gamma(\epsilon)]\left|\mathcal{K}_{\beta,1}(\mathcal{L})\right|}{\left|\mathcal{K}_{\beta,1}(\mathcal{L})\right|^{2\alpha}+[\gamma(\epsilon)]}\frac{\big\langle\mathcal{F},e_k\big\rangle}{\left|\mathcal{K}_{\beta,1}(\mathcal{L})\right|^{\theta+1}}e_k(x)\right\|_2^{\frac{1}{\theta+1}}\\
\le&\left\|\sum_{k=1}^{+\infty}\frac{[\gamma(\epsilon)]}{\left|\mathcal{K}_{\beta,1}(\mathcal{L})\right|^{2\alpha}+[\gamma(\epsilon)]}\big\langle\rho,e_k\big\rangle e_k(x)\right\|_2^{\frac{\theta}{\theta+1}}\\
&\times\left\|\sum_{k=1}^{+\infty}\frac{[\gamma(\epsilon)]}{\left|\mathcal{K}_{\beta,1}(\mathcal{L})\right|^{2\alpha}+[\gamma(\epsilon)]}\frac{\big\langle\mathcal{F},e_k\big\rangle}{\left|\mathcal{K}_{\beta,1}(\mathcal{L})\right|^{\theta}}e_k(x)\right\|_2^{\frac{1}{\theta+1}}
\end{aligned} \tag{5.45}$$

From (5.45), we know

$$\begin{aligned}
I_4\le&\left(\left\|\sum_{n=1}^{\infty}\frac{[\gamma(\epsilon)]}{\left|\mathcal{K}_{\beta,1}(\mathcal{L})\right|^{2\alpha}+[\gamma(\epsilon)]}\big\langle\rho-\rho^{\epsilon},e_k\big\rangle e_k(x)\right\|_2\right.\\
&\left.+\left\|\sum_{n=1}^{\infty}\frac{[\gamma(\epsilon)]}{\left|\mathcal{K}_{\beta,1}(\mathcal{L})\right|^{2\alpha}+[\gamma(\epsilon)]}\big\langle\rho^{\epsilon},e_k\big\rangle e_k(x)\right\|_2\right)^{\frac{\theta}{\theta+1}}\left\|\sum_{k=1}^{+\infty}\left(\frac{\lambda_n}{\mathcal{C}_\beta}\right)^{\theta}\big\langle f,e_k\big\rangle e_k(x)\right\|_2^{\frac{1}{\theta+1}}\\
\le&(\epsilon+\tau\epsilon)^{\frac{\theta}{\theta+1}}\mathcal{C}_\beta^{-\frac{\theta}{\theta+1}}E^{\frac{1}{\theta+1}}=\left(\frac{\tau+1}{\mathcal{C}_\beta}\right)^{\frac{\theta}{\theta+1}}E^{\frac{1}{\theta+1}}\epsilon^{\frac{\theta}{\theta+1}}.
\end{aligned} \tag{5.46}$$

Combining (5.44)-(5.46), we obtain

$$\left\|\mathcal{F}^{\epsilon}_{[\gamma(\epsilon)]}-\mathcal{F}\right\|_2\le\left(\mathcal{A}_1\left(\frac{\mathcal{A}_4}{\tau-1}\right)^{\frac{1}{\theta+1}}+\left(\frac{\tau+1}{\mathcal{C}_\beta}\right)^{\frac{\theta}{\theta+1}}\right)E^{\frac{1}{\theta+1}}\epsilon^{\frac{\theta}{\theta+1}}. \tag{5.47}$$

- For $\theta\ge 2\alpha-1$, estimate of $\mathcal{I}_3$, and Lemma 5.3, we have

$$\mathcal{I}_3=\left\|\mathcal{F}_[\gamma(\epsilon)]^{\epsilon}-\mathcal{F}_{[\gamma(\epsilon)]}\right\|_2\le\mathcal{A}_1\epsilon[\gamma(\epsilon)]^{-\frac{1}{2\alpha}}\le\left(\frac{\mathcal{A}_5}{\tau-1}\right)^{\frac{1}{2\alpha}}E^{\frac{1}{2\alpha}}\epsilon^{\frac{2\alpha-1}{2\alpha}}. \tag{5.48}$$

Estimate of $\mathcal{I}_4$, by Lemma 5.3, we know that

$$\begin{aligned}\mathcal{I}_4 &= \big\|\mathcal{F}-\mathcal{F}_{[\gamma(\epsilon)]}\big\|_2 = \Big\|\sum_{k=1}^{+\infty}\frac{[\gamma(\epsilon)]}{\left(T^\alpha E_{\alpha,1+\alpha}\left(-\lambda_n T^\alpha\right)\right)^{2\gamma}+[\gamma(\epsilon)]}\big\langle\mathcal{F},e_k\big\rangle e_k(x)\Big\|_2\\ &=\Big\|\sum_{k=1}^{+\infty}\frac{[\gamma(\epsilon)]\mathcal{K}_{\beta,1}(\mathcal{L})}{\big|\mathcal{K}_{\beta,1}(\mathcal{L})\big|^{2\alpha}+[\gamma(\epsilon)]}\frac{\big\langle\mathcal{F},e_k\big\rangle}{\big|\mathcal{K}_{\beta,1}(\mathcal{L})\big|}e_k(x)\Big\|_2\\ &\le\left\|\sum_{k=1}^{+\infty}\frac{[\gamma(\epsilon)]\big|\mathcal{K}_{\beta,1}(\mathcal{L})\big|}{\big|\mathcal{K}_{\beta,1}(\mathcal{L})\big|^{2\alpha}+[\gamma(\epsilon)]}\big\langle\mathcal{F},e_k\big\rangle e_k(x)\right\|_2^{1-\frac{1}{2\alpha}}\times\left\|\sum_{k=1}^{+\infty}\frac{[\gamma(\epsilon)]\big|\mathcal{K}_{\beta,1}(\mathcal{L})\big|}{\big|\mathcal{K}_{\beta,1}(\mathcal{L})\big|^{2\alpha}+[\gamma(\epsilon)]}\frac{\big\langle\mathcal{F},e_k\big\rangle}{\big|\mathcal{K}_{\beta,1}(\mathcal{L})\big|^{2\alpha}}\right\|_2^{\frac{1}{2\alpha}}\\ &\le\Big\|\sum_{k=1}^{+\infty}\frac{[\gamma(\epsilon)]}{\big|\mathcal{K}_{\beta,1}(\mathcal{L})\big|^{2\alpha}+[\gamma(\epsilon)]}\big\langle\rho,e_k\big\rangle e_k(x)\Big\|_2^{1-\frac{1}{2\alpha}}\times\Big\|\sum_{k=1}^{+\infty}\frac{[\gamma(\epsilon)]}{\big|\mathcal{K}_{\beta,1}(\mathcal{L})\big|^{2\alpha}+[\gamma(\epsilon)]}\frac{\big\langle\mathcal{F},e_k\big\rangle}{\big|\mathcal{K}_{\beta,1}(\mathcal{L})\big|^{2\alpha-1}}e_k(x)\Big\|_2^{\frac{1}{2\gamma}}.\end{aligned} \tag{5.49}$$

This leads to

$$\begin{aligned}\mathcal{I}_4 &\le\Bigg(\Big\|\sum_{k=1}^{+\infty}\frac{[\gamma(\epsilon)]}{|\mathcal{K}_{\beta,1}(\mathcal{L})|^{2\alpha}+[\gamma(\epsilon)]}\big\langle\rho-\rho^\epsilon,e_k\big\rangle e_k(x)\Big\|_2\\ &+\Big\|\sum_{k=1}^{+\infty}\frac{[\gamma(\epsilon)]}{\big|\mathcal{K}_{\beta,1}(\mathcal{L})\big|^{2\alpha}+[\gamma(\epsilon)]}\big\langle\rho_\epsilon,e_k\big\rangle e_k(x)\Big\|_2\Bigg)^{1-\frac{1}{2\alpha}}\times\Big\|\sum_{k=1}^{+\infty}\left(\frac{\lambda_k}{\mathcal{C}_\beta}\right)^{2\gamma-1}\lambda_k^{-\theta}\lambda_k^{\theta}\langle\mathcal{F},e_k\rangle e_k(x)\Big\|_2^{\frac{1}{2\alpha}}\\ &\le(\epsilon+\tau\epsilon)^{1-\frac{1}{2\alpha}}\underline{C}^{-\left(1-\frac{1}{2\alpha}\right)}\lambda_k^{2\alpha-1-\theta}E^{\frac{1}{2\alpha}}=\left(\frac{\tau+1}{\mathcal{C}_\beta}\right)^{1-\frac{1}{2\alpha}}\lambda_1^{2\alpha-1-\theta}E^{\frac{1}{2\alpha}}\epsilon^{1-\frac{1}{2\alpha}}.\end{aligned} \tag{5.50}$$

Combining (5.48) and (5.50), we obtain

$$\big\|\mathcal{F}^\epsilon_{[\gamma(\epsilon)]}-\mathcal{F}\big\|_2\le\left(\left(\frac{\mathcal{A}_5}{\tau-1}\right)^{\frac{1}{2\alpha}}+\left(\frac{\tau+1}{\mathcal{C}_\beta}\right)^{1-\frac{1}{2\alpha}}\lambda_1^{2\alpha-1-\theta}\right)E^{\frac{1}{2\alpha}}\epsilon^{1-\frac{1}{2\alpha}}. \tag{5.51}$$

This completes the proof. □

6. Discrete random noise

Lemma 6.1. *Let* $p=1,\ldots,n-1$, *and* $q=1,2,\ldots$, *with* $x_k=\pi\dfrac{2k-1}{2n}$ *and* $e_p(x_k)=\sqrt{\dfrac{2}{\pi}}\sin(px_k)$, *then we have*

$$\mathcal{S}_{p,q}=\frac{1}{n}\sum_{k=1}^{n}e_p(x_k)e_q(x_k)=\begin{cases}\dfrac{1}{\pi}, & q-p=2\ell n \text{ or } q+p=2\ell n(\ell \text{ even }),\\ -\dfrac{1}{\pi}, & q-p=2\ell n \text{ or } q+p=2\ell n(\ell \text{ odd }),\\ 0, & \text{otherwise.}\end{cases} \tag{6.52}$$

If $q=1,2,\ldots,n-1$, *then*

$$\mathcal{S}_{p,q}=\begin{cases}\dfrac{1}{\pi}, & p=q,\\ 0, & p\neq q,\end{cases}\text{, and }\ \frac{1}{n}\sum_{k=1}^{n}e_p(x_k)=\begin{cases}0, & p\neq 2\ell n,\\ (-1)^\ell\sqrt{\dfrac{2}{\pi}}, & p=2\ell n.\end{cases} \tag{6.53}$$

Lemma 6.2. *(See [15]) Let $k, m \in \mathbb{N}$ such that $1 \le k \le m-1$, and $\rho \in C[0,\pi]$. Then we have*

$$\langle \rho, e_k \rangle = \frac{\pi}{m} \sum_{k=1}^{m} \rho(x_k) e_k(x_k) - \sum_{\ell=1}^{\infty} (-1)^{\ell} \big(\langle \rho, e_{k+2\ell m} \rangle + \langle \rho e_{-k+2\ell m} \rangle \big), \quad 1 \le k \le m-1, \tag{6.54}$$

Lemma 6.3. *Let $0 < M_{tr} < m, M_{tr} \in \mathbb{N}$, assume that ρ is as in Lemma 6.2, then the source function $\mathcal{F}$ is given by*

$$\begin{aligned} \mathcal{F}_{m,M_{tr}}(x) = \sum_{k=1}^{M_{tr}} \frac{-k^2}{1 - E_{\beta,1}(-\mathcal{L}^{\beta} k)} \Bigg(\frac{\pi}{m} \sum_{p=1}^{m} \rho(x_p) e_k(x_p) \\ - \sum_{\ell=1}^{\infty} (-1)^{\ell} \Big(\langle \rho, e_{k+2\ell m} \rangle + \langle \rho, e_{-k+2\ell m} \rangle \Big) \Bigg) e_k(x) \\ + \sum_{k=M_{tr}+1}^{\infty} \frac{-k^2}{1 - E_{\beta,1}(-\mathcal{L}^{\beta} k)} \langle \rho, e_k \rangle e_k(x). \end{aligned} \tag{6.55}$$

7. The main results

Theorem 7.1. *Let $\epsilon > 0$ and $\epsilon_k \sim N(0,1)$ be independent normal random variables with $p = 1, \cdots, n$ (as mentioned above), then a regularized function $\widetilde{\mathcal{F}}_{m,M_{tr}}$ for $\mathcal{F}$ can be computed as follows*

$$\widetilde{\mathcal{F}}_{m,M_{tr}}(x) = \sum_{k=1}^{M_{tr}} \Big(\frac{-k^2}{1 - E_{\beta,1}(-\mathcal{L}^{\beta} k)} \Big) \frac{\pi}{m} \sum_{p=1}^{m} \rho(x_p) e_k(x_p) e_k(x). \tag{7.56}$$

M_{tr} is regularization parameters, it gives

$$\mathbb{E}\big\| \widetilde{\mathcal{F}}_{m,\mathcal{M}_{tr}} - \mathcal{F} \big\|_2 \le \sqrt{2} \big(M_{tr} \big)^{-\sigma} E + \frac{2\sqrt{2} M_{tr}^2}{1 - E_{\beta,1}(-\mathcal{L}^{\beta})} \left(\frac{\pi^2}{m^2} \mathcal{V}_{\max}^2 + \frac{\pi^4}{144} \frac{\|\mathcal{F}\|_2^2}{m^4} \right)^{\frac{1}{2}}. \tag{7.57}$$

Let M_{tr} such that $0 < M_{tr} < m$ and $\lim_{m \to +\infty} \dfrac{M_{tr}^2}{m} = 0$

$$\mathbb{E}\big\| \widetilde{\mathcal{F}}_{m,M_{tr}} - \mathcal{F} \big\|_2 \ \textbf{\textit{is of order}} \ \left\{ \frac{M_{tr}^2}{m}, (M_{tr})^{-\sigma} \right\}. \tag{7.58}$$

Remark 7.1. *By choosing $M_{tr} = m^{\frac{1}{2+\sigma}}$, then we have*

$$\mathbb{E}\big\| \widetilde{\mathcal{F}}_{m,\mathcal{M}_{tr}} - \mathcal{F} \big\|_2 \ \textbf{\textit{is of order}} \ m^{-\frac{\sigma}{2+\sigma}}. \tag{7.59}$$

Proof. It is easy to see that

$$|\langle \rho, e_k \rangle|^2 = \Big(\frac{1 - E_{\beta,1}(-\mathcal{L}^{\beta} k)}{k^2} \Big)^2 |\langle \mathcal{F}, e_k \rangle|^2 \le \frac{\|\mathcal{F}\|_2^2}{k^2}. \tag{7.60}$$

Using (7.56), we obtain

$$\begin{aligned} &\widetilde{\mathcal{F}}_{m,M_{tr}}(x) - \mathcal{F}(x) \\ &= \sum_{k=1}^{M_{tr}} \Big(\frac{k^2}{1 - E_{\beta,1}(-\mathcal{L}^{\beta} k)} \Big) \Big(\frac{\pi}{m} \sum_{k=1}^{m} \sigma_k \epsilon_k e_k(x_k) + \sum_{\ell=1}^{\infty} (-1)^{\ell} \Big(\langle \rho, e_{k+2\ell m} \rangle + \langle \rho, e_{-k+2\ell m} \rangle \Big) \Big) e_k(x) \\ &- \sum_{k=M_{tr}+1}^{\infty} \Big(\frac{k^2}{1 - E_{\beta,1}(-\mathcal{L}^{\beta} k)} \Big) \langle \rho, e_k \rangle \, e_k(x). \end{aligned} \tag{7.61}$$

Fom (7.61), thank to Lemma 6.3, we have

$$
\begin{aligned}
&\left\|\widetilde{\mathcal{F}}_{m,M_{tr}}-\mathcal{F}\right\|_2^2\\
&=\sum_{k=1}^{M_{tr}}\Big(\frac{k^2}{1-E_{\beta,1}(-\mathcal{L}^\beta k)}\Big)^2\Big(\frac{\pi}{m}\sum_{k=1}^{m}\sigma_k\epsilon_k e_k(x_k)+\sum_{\ell=1}^{\infty}(-1)^\ell\Big(\langle\rho,e_{k+2\ell m}\rangle+\langle\rho,e_{-k+2\ell m}\rangle\Big)\Big)^2\\
&+\sum_{k=M_{tr}+1}^{\infty}\Big(\frac{k^2}{1-E_{\beta,1}(-\mathcal{L}^\beta k)}\Big)^2\big|\langle\rho,e_k\rangle\big|^2. \qquad (7.62)
\end{aligned}
$$

The fact that $\mathbb{E}\left(\epsilon_j\epsilon_l\right)=0;(j\neq l)$, and $\mathbb{E}\epsilon_j=0; j=1,2,\ldots,n$. One has

$$
\begin{aligned}
&\mathbb{E}\left\|\widetilde{\mathcal{F}}_{n,\mathcal{M}_{tr}}-\mathcal{F}\right\|_{L^2(\Omega)}^2\\
&\le 2\sum_{k=M_{tr}+1}^{\infty}\Big(\frac{k^2}{1-E_{\beta,1}(-\mathcal{L}^\beta k)}\Big)^2\big|\langle\rho,e_k\rangle\big|^2\\
&+4\sum_{k=1}^{M_{tr}}\Big(\frac{k^2}{1-E_{\beta,1}(-\mathcal{L}^\beta k)}\Big)^2\Big(\frac{\pi^2}{m^2}\sum_{k=1}^{m}\sigma_k^2\mathbb{E}\epsilon_k^2+\Big(\sum_{\ell=1}^{\infty}(-1)^\ell\Big(\langle\rho,e_{k+2\ell m}\rangle+\langle\rho,e_{-k+2\ell m}\rangle\Big)\Big)^2\Big)\\
&\le\mathcal{J}_1+\mathcal{J}_2. \qquad (7.63)
\end{aligned}
$$

Of course $\sum_{\ell=1}^{\infty}\frac{1}{\ell^2}=\frac{\pi^2}{6}$, we obtain

$$
\sum_{\ell=1}^{\infty}\left|\langle\rho,e_{k+2\ell m}\rangle+\langle\rho,e_{-k+2\ell m}\rangle\right|\le\frac{\|\mathcal{F}\|_2}{m^2}\left[\sum_{\ell=1}^{\infty}\frac{1}{(k+2\ell m)^2}+\sum_{\ell=1}^{\infty}\frac{1}{(-k+2\ell m)^2}\right]\le\frac{\pi^2}{12}\frac{\|\mathcal{F}\|_2}{m^2}. \qquad (7.64)
$$

For $k\ge1$, $1=k^{2\sigma}k^{-2\sigma}$, estimate of $\mathcal{J}_1$

$$
\mathcal{J}_1^2=\sum_{k=M_{tr}+1}^{\infty}\Big(\frac{k^2}{1-E_{\beta,1}(-\mathcal{L}^\beta k)}\Big)^2\big|\langle\rho,e_k\rangle\big|^2=\sum_{k=M_{tr}+1}^{\infty}k^{2\sigma}k^{-2\sigma}\big|\langle\mathcal{F},e_k\rangle\big|^2, \qquad (7.65)
$$

this leads to

$$
\mathcal{J}_1^2\le\big(M_{tr}\big)^{-2\sigma}E^2. \qquad (7.66)
$$

Next, estimate of $\mathcal{J}_2$

$$
\mathcal{J}_2^2\le\sum_{k=1}^{M_{tr}}\Big(\frac{k^2}{1-E_{\beta,1}(-\mathcal{L}^\beta k)}\Big)^2\left(\frac{\pi^2}{m^2}\sum_{k=1}^{m}\sigma_k^2\mathbb{E}\epsilon_k^2+\frac{\pi^2}{12}\frac{\|\mathcal{F}\|_2}{m^2}\right)^2. \qquad (7.67)
$$

Since $\sigma_k\le\mathcal{V}_{\max}$, it gives

$$
\begin{aligned}
\mathcal{J}_2^2&\le\sum_{k=1}^{M_{tr}}\frac{4k^4}{\big(1-E_{\beta,1}(-\mathcal{L}^\beta)\big)^2}\Big(\frac{\pi^2}{n^2}\mathcal{V}_{\max}^2+\frac{\pi^4}{144}\frac{\|\mathcal{F}\|_2^2}{m^4}\Big)\\
&\le\frac{4M_{tr}^4}{\big(1-E_{\beta,1}(-\mathcal{L}^\beta)\big)^2}\Big(\frac{\pi^2}{m^2}\mathcal{V}_{\max}^2+\frac{\pi^4}{144}\frac{\|\mathcal{F}\|_2^2}{m^4}\Big). \qquad (7.68)
\end{aligned}
$$

Combining (7.63), (7.66), and (7.67), we obtain

$$
\mathbb{E}\left\|\widetilde{\mathcal{F}}_{m,M_{tr}}-\mathcal{F}\right\|_2^2\le2\big(M_{tr}\big)^{-2\sigma}E^2+\frac{8M_{tr}^4}{\big(1-E_{\beta,1}(-\mathcal{L}^\beta)\big)^2}\Big(\frac{\pi^2}{m^2}\mathcal{V}_{\max}^2+\frac{\pi^4}{144}\frac{\|\mathcal{F}\|_2^2}{m^4}\Big). \qquad (7.69)
$$

This completes the proof. □

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References

[1] R.S. Adiguzel, U. Aksoy, E. Karapinar, I.M. Erhan, On The Solutions Of Fractional Differential Equations Via Geraghty Type Hybrid Contractions, Appl. Comput. Math., 20, No 2, (2021),313-333.

[2] E. Karapinar, H. D. Binh, N. H. Luc, N. H. Can, On continuity of the fractional derivative of the time-fractional semilinear pseudo-parabolic systems, Advances in Difference Equations (2021) 2021:70

[3] R.S. Adiguzel, U. Aksoy, E. Karapinar, I.M. Erhan, Uniqueness of solution for higher-order nonlinear fractional differential equations with multi-point and integral boundary conditions, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas 115, no. 3 (2021): 1-16.

[4] M. Benchohra, E. Karapınar, J.E. Lazreg, & A. Salim, Impulsive Fractional Differential Equations with Retardation and Anticipation. In Fractional Differential Equations: New Advancements for Generalized Fractional Derivatives (2023), 109-155.

[5] A. Benaissa, A. Salim, M. Benchohra, E. Karapınar, Functional delay random semilinear differential equations. The Journal of Analysis, 31(4), (2023),2675-2686.

[6] E. Karapınar, M. Cvetković, An inevitable note on bipolar metric spaces. AIMS Mathematics, 9(2), (2024), 3320-3331.

[7] H. Afshari, E. Karapınar, A discussion on the existence of positive solutions of the boundary value problems via-Hilfer fractional derivative on b-metric spaces, Advances in Difference Equations volume 2020, Article number: 616 (2020)

[8] H. Afshari, S. Kalantari, E. Karapinar, Solution of fractional differential equations via coupled fixed point, Electronic Journal of Differential Equations, (2015), No. 286, pp. 1-12

[9] B. Alqahtani, H. Aydi, E. Karapınar, V. Rakocevic, A Solution for Volterra Fractional Integral Equations by Hybrid Contractions, Mathematics (2019), 7, 694.

[10] G. Jothilakshmi, B.S. Vadivoo, Y. Almalki, A. Debbouche, Controllability analysis of multiple fractional order integro-differential damping systems with impulsive interpretation, Journal of Computational and Applied Mathematics, (2022), p.114204.

[11] A. Debbouche, Time-partial differential equations: Modeling and simulation, Mathematical Methods in the Applied Sciences 44, no. 15 (2021), 11767-11767.

[12] S. Guechi, D. Rajesh, D. Amar, M. Muslim, Analysis and Optimal Control of ϕ-Hilfer Fractional Semilinear Equations Involving Nonlocal Impulsive Conditions, Symmetry 13, no. 11 (2021): 2084.

[13] M.S.E. Meziani, S. Djemoui, N. Boussetila, Detection of source term in an abstract fractional subdivision model by the modified quasi-boundary value methoe with a priori and a posteriori estimate, to appear

[14] R. Sassane, N. Boussetila, F. Rebbani, A. Benrabah, Iterative regularization method for an abstract ill-posed generalized elliptic equation, Asian-European Journal of Mathematics, Vol. 14, No. 5 2150069 (2021), (22 pages).

[15] N. H. Tuan, E. Nane, Inverse source problem for time-fractional diffusion with discrete random noise, Statistics and Probability Letters, (2016),http://dx.doi.org/10.1016/j.spl.2016.09.026.

[16] X. Xiong, & X. Xue, A fractional Tikhonov regularization method for identifying a space-dependent source in the time-fractional diffusion equation. Applied Mathematics and Computation, 349, (2019), 292-303.

[17] N. D. Phuong, N. H. Luc, L. D. Long, *Modified Quasi Boundary Value method for inverse source problem of the bi-parabolic equation* , Advances in the Theory of Nonlinear Analysis and its Applications 4, **No. 3**, (2020),132–142.

[18] X. X. Li, J. L. Lei, & F. Yang, An a posteriori Fourier regularization method for identifying the unknown source of the space-fractional diffusion equation. Journal of Inequalities and Applications, (1), (2014), 1-13.

[19] F. Yang, Y. P. Ren, X. X. Li, & D. G. Li, . Landweber iterative method for identifying a space-dependent source for the time-fractional diffusion equation. Boundary Value Problems, (1), (2017), 1-19.

[20] N.D. Phuong, *Note on a Allen-Cahn equation with Caputo-Fabrizio derivative*, Results in Nonlinear Analysis 4 (2021) **No. 3**, 179–185.

[21] F. Yang, F. P. Ren, X.X. Li, The quasi-reversibility method for a final value problem of the time-fractional diffusion equation with inhomogeneous source. Mathematical Methods in the Applied Sciences.41(5), (2018), 1774-1795.

[22] J. Kokila, M. T. Nair, Fourier truncation method for the non-homogeneous time fractional backward heat conduction problem. Inverse Problems in Science and Engineering. 1-25. https://doi.org/10.1080/17415977.2019.1580707 (2019).

[23] X. B. Yan, T. Wei, Determine a Space-Dependent Source Term in a Time Fractional Diffusion-Wave Equation. Acta Appl. Math. 165, (2020), 163-181.